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Smoothing transformation and spline collocation for nonlinear fractional initial and boundary value problems☆

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ABSTRACT

We construct and justify a class of high order methods for the numerical solution of initial and boundary value problems for nonlinear fractional differential equations of the form $(D_*^\alpha y)(t) = f(t, y(t))$ with Caputo type fractional derivatives $D_*^\alpha y$ of order $\alpha > 0$. Using an integral equation reformulation of the underlying problem we first regularize the solution by a suitable smoothing transformation. After that we solve the transformed equation by a piecewise polynomial collocation method on a mildly graded or uniform grid. Optimal global convergence estimates are derived and a superconvergence result for a special choice of collocation parameters is established. To illustrate the reliability of the proposed algorithms two numerical examples are given.

1. Introduction

In the present paper we study the convergence behavior of a modified spline collocation method for the numerical solution of fractional initial and boundary value problems of the form

$$(D_*^\alpha y)(t) = f(t, y(t)), \quad 0 \le t \le b, \tag{1.1}$$

$$\sum_{j=0}^{n_0} \beta_{ij0}\, y^{(j)}(0) + \sum_{k=1}^{l} \sum_{j=0}^{n_1} \beta_{ijk}\, y^{(j)}(b_k) = \gamma_i, \quad i = 0, \dots, n-1, \ n := \lceil \alpha \rceil, \tag{1.2}$$

where $\beta_{ij0}, \beta_{ijk}, \gamma_i \in \mathbb{R} := (-\infty, \infty)$, $\lceil \alpha \rceil$ is the smallest integer greater than or equal to $\alpha \in \mathbb{R}$,

$$\begin{gathered} n-1 < \alpha < n, \qquad 0 < b_1 < \cdots < b_l \le b, \\ l, n \in \mathbb{N} := \{1, 2, \dots\}, \qquad n_0, n_1 \in \mathbb{N}_0 := \{0\} \cup \mathbb{N}, \quad n_0 < n, \ n_1 < n, \end{gathered} \tag{1.3}$$

$f : [0, b] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, and $D_*^\alpha y$ is the Caputo derivative of order $\alpha > 0$ of an unknown function y.

Observe that a special case of problem (1.1)–(1.2) is the following initial value problem:

$$\begin{gathered} (D_*^\alpha y)(t) = f(t, y(t)), \quad 0 \le t \le b, \ \alpha > 0, \\ y^{(i)}(0) = \gamma_i, \quad i = 0, \dots, n-1, \ n = \lceil \alpha \rceil. \end{gathered} \tag{1.4}$$

Thus, the results of this paper can be applied also to initial value problems of the form (1.4).

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The Caputo differentiation operator D_*^α of order $\alpha > 0$ can be defined by the formula (see, e.g, [1])

$$(D_*^\alpha y)(t) := (D^\alpha (y - \mathcal{Q}_{n-1}[y]))(t), \quad t > 0,\ n = \lceil \alpha \rceil,$$

where

$$\mathcal{Q}_{n-1}[y](s) := \sum_{i=0}^{n-1} \frac{y^{(i)}(0)}{i!} s^i$$

and D^α is the Riemann–Liouville fractional differentiation operator of order $\alpha > 0$:

$$(D^\alpha y)(t) := \frac{d^n}{dt^n}(J^{n-\alpha} y)(t), \quad t > 0,\ n = \lceil \alpha \rceil.$$

Here J^α, the Riemann–Liouville integral operator of order $\alpha > 0$, is defined by the formula

$$(J^\alpha y)(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\, ds, \quad t > 0, \tag{1.5}$$

where Γ is the Euler gamma function. For $\alpha = 0$ we set $D^0 = D_*^0 = J^0 := I$ where I is the identity mapping. If $\alpha = n \in \mathbb{N}$ then $D^n y = D_*^n y = y^{(n)}$ where $y^{(n)}$ is the usual nth order derivative of y.

The integral operator J^α, $\alpha > 0$, is linear, bounded and compact as an operator from $L^\infty(0, b)$ into $C[0, b]$ (see, e.g., [2]). Moreover, we have for any $y \in L^\infty(0, b)$ that (see, e.g., [3])

$$(J^\alpha y)^{(k)} \in C[0, b], \quad (J^\alpha y)^{(k)}(0) = 0,\ \alpha > 0,\ k = 0, \ldots, \lceil \alpha \rceil - 1, \tag{1.6}$$

$$J^\alpha J^\beta y = J^{\alpha+\beta} y, \quad \alpha > 0,\ \beta > 0, \tag{1.7}$$

$$D^\beta J^\alpha y = D_*^\beta J^\alpha y = J^{\alpha-\beta} y, \quad 0 < \beta \le \alpha. \tag{1.8}$$

Differential equations involving differential operators of fractional (non-integer) order have been proved to be a valuable tool in modeling many phenomena in the fields of physics, mechanics, chemistry, engineering, finance and others (see, for example, [4–6]). Mathematical aspects of fractional differential equations and methods of their solution have been discussed by many authors (see [1,3,7] and the references cited therein). Various existence and uniqueness results for initial value problems (1.4) can be found in [1] and for some boundary value problems of the form (1.1)–(1.2) in [8].

For most fractional differential equations we cannot provide methods to compute their solutions analytically and it is necessary to use numerical methods. Efficient numerical methods for solving integral and differential equations are spline collocation methods. These methods are also applicable for the numerical solution of fractional differential equations since initial and boundary value problems for fractional differential equations can be reformulated as certain integral equations of the second kind with weakly singular kernels (see, e.g., [1]). The convergence of spline collocation methods for weakly singular integral equations is analyzed in [9–11] and for fractional differential equations in [12–18]. A review of spectral collocation methods for solving fractional differential equations is given in [19].

From Theorem 2.1 we see that the derivatives of the solution of problem (1.1)–(1.2) and the corresponding integral equation may be unbounded only near the initial point 0 of the interval of integration $[0, b]$. Due to the lack of smoothness of the exact solution, piecewise polynomial collocation methods based on uniform grid for solving such equations show slow convergence behavior [20]. In order to construct methods with higher convergence order it is necessary to take into account the possible singular behavior of the exact solution. In particular, we can construct high order collocation methods by using polynomial splines on special graded grids where the grid points are more densely clustered near the singular point $t = 0$ of the exact solution $y(t)$ of (1.1)–(1.2) [13–16]. However, the use of strongly non-uniform grids may cause an accumulation of rounding errors and unstable behavior of numerical results. To diminish loss of precision we may resort to collocation on the uniform grid using non-polynomial basic functions which reflect the singular behavior of the exact solution. This approach has been applied for solving Volterra integral equations in [21–23] and for solving fractional differential equations in [24–26].

In the present paper we use an alternative approach for diminishing loss of precision. First we introduce an integral equation reformulation of problem (1.1)–(1.2). Then we perform in the integral equation a smoothing change of variables $t = b^{1-\rho}\tau^\rho$ ($\rho \in [1, \infty)$) so that the singularities of the (usual) derivatives of the exact solution will be milder or disappear. After that we solve the transformed integral equation by a piecewise polynomial collocation method on a mildly graded or uniform grid. The final step of our method is based on a conversion of the obtained spline functions into the (typically non-polynomial) approximate solution for (1.1)–(1.2). Similar ideas for solving Volterra integral equations are used in [27–31] and for solving fractional differential equations in [32–35].

The rest of our paper is arranged as follows. In Section 2 some a priori estimates for higher order derivatives of the exact solution of problem (1.1)–(1.2) are presented (see Theorem 2.1). These estimates will play a key role in the convergence analysis of the proposed algorithms in Sections 5 and 6. In Section 3 some results about piecewise polynomial interpolation on graded grids are presented. In Section 4 a class of smoothing transformations is introduced and based on these transformations and spline collocation techniques a class of numerical methods for solving (1.1)–(1.2) is constructed.

In Sections 5 and 6 the attainable order of convergence of proposed algorithms is studied. Optimal global convergence estimates are derived and a superconvergence result for a special choice of collocation parameters is established. Finally, in Section 7 the obtained theoretical results are tested by two numerical examples.

The main results of the present paper are formulated in Theorems 2.1, 5.1 and 6.1.

2. Smoothness of the solution

Using some ideas from [14] (see also [1,16]) we find first an integral equation reformulation for the problem (1.1)–(1.2). Let $y \in C[0,b]$ be such that $D_*^\alpha y \in C[0,b]$. Introduce a new unknown function $z := D_*^\alpha y$. Then (see [1,3])

$$y(t) = (J^\alpha z)(t) + \sum_{\lambda=0}^{n-1} c_\lambda t^\lambda, \quad t \in [0,b], \ n = \lceil \alpha \rceil \in \mathbb{N}, \tag{2.1}$$

where $c_\lambda \in \mathbb{R}$ $(\lambda = 0, \dots, n-1)$ are arbitrary constants. The function y of the form (2.1) satisfies the boundary value conditions (1.2) if and only if (see (1.8) and (1.6))

$$\sum_{j=0}^{n_0} \beta_{ij0}\, j!\, c_j + \sum_{k=1}^{l}\sum_{j=0}^{n_1} \beta_{ijk}\left[(J^{\alpha-j}z)(b_k) + \sum_{\lambda=j}^{n-1} \frac{\lambda!}{(\lambda-j)!} b_k^{\lambda-j} c_\lambda\right] = \gamma_i, \quad i = 0, \dots, n-1.$$

We rewrite these conditions in the form

$$\sum_{j=0}^{n-1}\left[j!\,\beta_{ij0} + \sum_{k=1}^{l}\sum_{\lambda=0}^{j} \beta_{i\lambda k} \frac{j!}{(j-\lambda)!} b_k^{j-\lambda}\right] c_j = \gamma_i - \sum_{k=1}^{l}\sum_{j=0}^{n_1} \beta_{ijk}(J^{\alpha-j}z)(b_k), \quad i = 0, \dots, n-1, \tag{2.2}$$

setting $\beta_{ij0} = 0$ for $j > n_0$ and $\beta_{ijk} = 0$ for $j > n_1$ $k = 1, \dots, l$. Clearly, (2.2) is a linear system of n equations with respect to $c_0, \dots, c_{n-1}$. Let

$$M := \left(j!\,\beta_{ij0} + \sum_{k=1}^{l}\sum_{\lambda=0}^{j} \beta_{i\lambda k} \frac{j!}{(j-\lambda)!} b_k^{j-\lambda}\right)_{i,j=0}^{n-1}$$

be the matrix of the system (2.2). In the sequel we assume that the matrix M is regular. Observe that M is regular if and only if from all polynomials y of degree $n-1$ only $y = 0$ satisfies the homogeneous boundary conditions

$$\sum_{j=0}^{n_0} \beta_{ij0} y^{(j)}(0) + \sum_{k=1}^{l}\sum_{j=0}^{n_1} \beta_{ijk} y^{(j)}(b_k) = 0, \quad i = 0, \dots, n-1, \tag{2.3}$$

corresponding to the conditions (1.2) with $\gamma_i = 0$, $i = 0, \dots, n-1$. Indeed, substituting (2.1) with $z = 0$ into (2.3) we obtain a homogeneous system with respect to $c_0, c_1, \dots, c_{n-1}$. This system coincides with the system (2.2) with $\gamma_i = 0$ $(i = 0, \dots, n-1)$ and $z = 0$. Therefore, corresponding to (2.2) homogeneous system has only the trivial solution $c_0 = c_1 = \cdots = c_{n-1} = 0$ (and thus M is regular) if and only if from all polynomials y of degree $n-1$ only $y = 0$ satisfies (2.3).

Let $M^{-1} = (p_{\lambda,\mu})_{\lambda,\mu=0}^{n-1}$ be the inverse of M. Using M^{-1}, the solution of the system (2.2) can be written in the form

$$c_\lambda = \delta_\lambda - \sum_{k=1}^{l}\sum_{j=0}^{n_1} \kappa_{\lambda jk}(J^{\alpha-j}z)(b_k), \quad \lambda = 0, \dots, n-1,$$

where

$$\delta_\lambda := \sum_{\mu=0}^{n-1} p_{\lambda\mu}\,\gamma_\mu, \qquad \kappa_{\lambda jk} := \sum_{\mu=0}^{n-1} p_{\lambda\mu}\,\beta_{\mu jk}. \tag{2.4}$$

Therefore a function y in the form (2.1) satisfies the conditions (1.2) if and only if it can be expressed by the formula

$$y = Gz + Q \tag{2.5}$$

where

$$(Gz)(t) := (J^\alpha z)(t) - \sum_{\lambda=0}^{n-1} t^\lambda \sum_{k=1}^{l}\sum_{j=0}^{n_1} \kappa_{\lambda jk}(J^{\alpha-j}z)(b_k), \quad t \in [0,b], \tag{2.6}$$

$$Q(t) := \sum_{\lambda=0}^{n-1} \delta_\lambda t^\lambda, \quad t \in [0,b]\,. \tag{2.7}$$

Remark 2.1. In the case of an initial value problem (1.4) we have

$$G = J^{\alpha}, \quad Q(t) = \sum_{\lambda=0}^{n-1} \frac{\gamma_\lambda}{\lambda!} t^{\lambda}.$$

Suppose now that $y^* \in C[0,b]$ is a solution of the boundary value problem (1.1)–(1.2) such that $z^* := D_*^{\alpha} y^* \in C[0,b]$. Then it follows from the observations above that y^* has the form (see (2.5)) $y^* = Gz^* + Q$ where G and Q are defined by the formulas (2.6) and (2.7), respectively. Inserting (2.5) into (1.1), we see that $z^* = D_*^{\alpha} y^*$ satisfies the equation

$$z(t) = f(t, (Gz)(t) + Q(t)), \quad t \in [0, b]. \tag{2.8}$$

Conversely, it is easy to show that if $z^* \in C[0,b]$ is a solution of Eq. (2.8) then $y^* = Gz^* + Q$ is a solution to (1.1)–(1.2). In this sense Eq. (2.8) is equivalent to the boundary value problem (1.1)–(1.2) and we can use it in construction of high order methods for the numerical solution of problem (1.1)–(1.2). Observe that (2.8) is a nonlinear integral equation with respect to z.

An important question that arises by studying the attainable order of the convergence of a numerical method is the smoothness of the exact solution of a fractional differential equation. Some information about the smoothness properties of the solution can be obtained by using asymptotic expansions of the solution in fractional powers with respect to the independent variable (see, e.g., [1]).

In the present paper we use another approach: in order to characterize the singularities of the derivatives of a solution of problem (1.1)–(1.2) at $t = 0$, we introduce a weighted space of smooth functions $C^{q,\nu}(0, b]$ (cf., e.g., [2]). For given $q \in \mathbb{N}$ and $\nu \in \mathbb{R}$, $\nu < 1$, by $C^{q,\nu}(0, b]$ we denote the set of continuous functions $y : [0, b] \to \mathbb{R}$ which are q times continuously differentiable in $(0, b]$ and such that for all $t \in (0, b]$ and $i = 1, \ldots, q$ the following estimate holds:

$$\left|y^{(i)}(t)\right| \le c \begin{Bmatrix} 1 & \text{if } i < 1-\nu \\ 1 + |\log t| & \text{if } i = 1-\nu \\ t^{1-\nu-i} & \text{if } i > 1-\nu \end{Bmatrix}.$$

Here $c = c(y)$ is a positive constant. Equipped with a suitable norm the set $C^{q,\nu}(0, b]$ becomes a Banach space (see, e.g., [2,35]). Clearly,

$$C^q[0, b] \subset C^{q,\nu}(0, b] \subset C^{m,\mu}(0, b] \subset C[0, b], \quad q \ge m \ge 1, \ \nu \le \mu < 1. \tag{2.9}$$

Note that a function of the form $y(t) = g_1(t)\, t^{\mu} + g_2(t)$ is included in $C^{q,\nu}(0, b]$ if $\mu \ge 1 - \nu > 0$ and $g_j \in C^q[0, b]$, $j = 1, 2$.

For the proof of Theorem 2.1 we need the following result from [15].

Lemma 2.1. *Assume that the following conditions for Eq. (1.1) are fulfilled:*

(i) $\alpha > 0$, $\alpha \notin \mathbb{N}$ *and* $f : \bar{\Omega} \to \mathbb{R}$ *is a continuous function which is q times ($q \in \mathbb{N}$) continuously differentiable in Ω where*

$$\Omega := \{(t, y) : t \in (0, b], y \in \mathbb{R}\}, \qquad \bar{\Omega} := \{(t, y) : t \in [0, b], y \in \mathbb{R}\}; \tag{2.10}$$

(ii) *there exist a monotone increasing function $\psi : [0, \infty) \to \mathbb{R}$ and a real number $\nu \in [1-\alpha, 1)$ such that for all nonnegative integers i and j with $i + j \le q$ and for all $(t, y) \in \Omega$*

$$\left|\frac{\partial^{i+j}}{\partial t^i \partial y^j} f(t, y)\right| \le \psi(|y|) \begin{Bmatrix} 1 & \text{if } i < 1-\nu \\ 1 + |\log t| & \text{if } i = 1-\nu \\ t^{1-\nu-i} & \text{if } i > 1-\nu \end{Bmatrix}; \tag{2.11}$$

if $\alpha \in (0, 1)$, then we assume in addition to (2.11) that for all nonnegative integers i and j with $i + j \le q$ and for all $(t, y_1), (t, y_2) \in \Omega$ it holds

$$\left|\frac{\partial^{i+j}}{\partial t^i \partial y^j}[f(t, y_1) - f(t, y_2)]\right| \le \psi(\max\{|y_1|, |y_2|\})|y_1 - y_2| \begin{Bmatrix} 1 & \text{if } i = 0 \\ t^{1-\nu-i} & \text{if } i > 0 \end{Bmatrix}; \tag{2.12}$$

(iii) *for arbitrary given constants $\delta_i \in \mathbb{R}$ ($i = 0, \ldots, n-1$) Eq. (1.1) possesses a solution $y^{**} \in C[0, b]$ such that $D_*^{\alpha} y^{**} \in C[0, b]$ and*

$$(y^{**})^{(i)}(0) = \delta_i, \quad i = 0, \ldots, n-1, \ n = \lceil \alpha \rceil.$$

Then $y^{**} \in C^{q,\nu}(0, b]$ and $D_*^{\alpha} y^{**} \in C^{q,\nu}(0, b]$.

The regularity of a solution to (1.1)–(1.2) can be characterized by following theorem.

Theorem 2.1. *Let the conditions* (i) *and* (ii) *of Lemma 2.1 be fulfilled. Moreover, assume that (1.3) is true and from all polynomials y of degree $\lceil \alpha \rceil - 1$ only $y = 0$ satisfies the conditions (2.3). Finally, suppose that the boundary value problem (1.1)–(1.2) possesses a solution $y^* \in C[0, b]$ such that $D_*^{\alpha} y^* \in C[0, b]$.*

Then $y^* \in C^{q,\nu}(0, b]$ and $D_*^{\alpha} y^* \in C^{q,\nu}(0, b]$.

Proof. Let $y^* \in C[0,b]$ be a solution to (1.1)–(1.2) such that $D_*^\alpha y^* \in C[0,b]$. Denote $z^* := D_*^\alpha y^*$ and

$$c_\lambda^* := \delta_\lambda - \sum_{k=1}^{l}\sum_{j=0}^{n-1} \kappa_{\lambda jk}(J^{\alpha-j}z^*)(b_k), \quad \lambda = 0, \dots, n-1, \ n = \lceil\alpha\rceil,$$

where δ_λ and $\kappa_{\lambda jk}$ are defined by (2.4). Then (see (2.1)),

$$y^*(t) = (J^\alpha z^*)(t) + \sum_{\lambda=0}^{n-1} c_\lambda^* t^\lambda, \quad t \in [0,b].$$

Consequently, y^* is a solution to Eq. (1.1) which satisfies the following initial conditions (see (1.8) and (1.6)):

$$(y^*)^{(i)}(0) = i!\, c_i^*, \quad i = 0, \dots, n-1.$$

This together with Lemma 2.1 yields the assertions of Theorem 2.1. □

3. Piecewise polynomial interpolation

For $N \in \mathbb{N}$ and $1 \le r < \infty$, let $\Pi_N := \{t_0, \dots, t_N\}$ be a partition (a graded grid) of the interval $[0,b]$ with the grid points

$$t_j = b\left(\frac{j}{N}\right)^r, \quad j = 0, 1, \dots, N, \tag{3.1}$$

where $r \in [1,\infty)$ is the so called grading exponent. If $r = 1$, then the grid points (3.1) are distributed uniformly; for $r > 1$ the grid points (3.1) are more densely clustered near the left endpoint of the interval $[0,b]$.

For a given integer $m \ge 0$ we denote by $S_m^{(-1)}(\Pi_N)$ the standard space of piecewise polynomial functions on $[0,b]$ associated with the grid Π_N:

$$S_m^{(-1)}(\Pi_N) = \left\{ v : v\big|_{(t_{j-1},t_j)} \in \pi_m,\ j = 1, \dots, N \right\}.$$

Here $v\big|_{(t_{j-1},t_j)}$ is the restriction of $v : [0,b] \to \mathbb{R}$ onto the subinterval (t_{j-1}, t_j) and π_m denotes the set of polynomials of degree not exceeding m. Note that the elements of $S_m^{(-1)}(\Pi_N)$ may have jump discontinuities at the interior points $t_1, \dots, t_{N-1}$ of the grid Π_N.

In every interval $[t_{j-1}, t_j]$, $j = 1, \dots, N$, we define $m \in \mathbb{N}$ collocation points $t_{j1}, \dots, t_{jm}$ by formula

$$t_{jk} = t_{j-1} + \eta_k(t_j - t_{j-1}), \quad k = 1, \dots, m,\ j = 1, \dots, N, \tag{3.2}$$

where $\eta_1 \dots, \eta_m$ are some fixed (collocation) parameters which do not depend on j and N and satisfy

$$0 \le \eta_1 < \eta_2 < \dots < \eta_m \le 1. \tag{3.3}$$

For given $N, m \in \mathbb{N}$ let $\mathcal{P}_N : C[0,b] \to S_{m-1}^{(-1)}(\Pi_N)$ be an interpolation operator such that

$$\mathcal{P}_N v \in S_{m-1}^{(-1)}(\Pi_N), \quad (\mathcal{P}_N v)(t_{jk}) = v(t_{jk}),\ k = 1, \dots, m,\ j = 1, \dots, N, \tag{3.4}$$

for any continuous function $v \in C[0,b]$. If $\eta_1 = 0$, then by $(\mathcal{P}_N v)(t_{j1})$ we denote the right limit $\lim_{t\to t_{j-1}, t>t_{j-1}} (\mathcal{P}_N v)(t)$. If $\eta_m = 1$, then by $(\mathcal{P}_N v)(t_{jm})$ we denote the left limit $\lim_{t\to t_j, t<t_j} (\mathcal{P}_N v)(t)$.

In what follows, for Banach spaces X and Y, by $\mathcal{L}(X,Y)$ is denoted the Banach space of linear bounded operators $A : X \to Y$ with the norm $\|A\|_{\mathcal{L}(X,Y)} := \sup\{\|Ax\|_Y : x \in X, \|x\|_X \le 1\}$. The following properties of $\mathcal{P}_N$ are well-known (see, e.g., [2]).

Lemma 3.1. *Let $\mathcal{P}_N : C[0,b] \to L^\infty(0,b)$ $(N \in \mathbb{N})$ be defined by (3.4). Then $\mathcal{P}_N \in \mathcal{L}(C[0,b], L^\infty(0,b))$ and the norms of $\mathcal{P}_N$ are uniformly bounded:*

$$\|\mathcal{P}_N\|_{\mathcal{L}(C[0,b],L^\infty(0,b))} \le c, \quad N \in \mathbb{N},$$

with a constant c which is independent of N. Moreover, we have

$$\|\mathcal{P}_N z - z\|_\infty \to 0 \quad \text{for every} \quad z \in C[0,b] \quad \text{as } N \to \infty, \tag{3.5}$$

where

$$\|v\|_\infty := \sup_{0<t<b} |v(t)|, \quad v \in L^\infty(0,b).$$

If $z \in C^{m,\nu}(0,b]$, $m \in \mathbb{N}$, $-\infty < \nu < 1$, *then*

$$\|\mathcal{P}_N z - z\|_\infty \le c \left\{ \begin{array}{lll} N^{-r(1-\nu)} & \text{for } 1 \le r < \dfrac{m}{1-\nu} \\ N^{-m}(1+\log N) & \text{for } r = \dfrac{m}{1-\nu} = 1 \\ N^{-m} & \text{for } r = \dfrac{m}{1-\nu} > 1 \text{ or } r > \dfrac{m}{1-\nu} \end{array} \right\}. \tag{3.6}$$

Here $r \in [1,\infty)$ *is introduced in* (3.1) *and* c *is a constant not depending on* N.

4. Smoothing transformation and numerical solutions

Let us consider a change of variables

$$t = b^{1-\rho}\tau^\rho, \quad \tau \in [0,b], \tag{4.1}$$

depending on a parameter $\rho \in [1,\infty)$. Then $\tau = b^{(\rho-1)/\rho}t^{1/\rho} \in [0,b]$ for $t \in [0,b]$. In the case $\rho = 1$ it follows from (4.1) that $t = \tau$. We are interested in transformations (4.1) with $\rho > 1$ since such transformations possess a smoothing property for $z \in C[0,b] \cap C^q(0,b]$ ($q \in \mathbb{N}$) with singularities of derivatives of $z(t)$ at $t = 0$. Actually, we obtain from [29,30] the following result.

Lemma 4.1. *Let* $z \in C^{q,\nu}(0,b]$ ($q \in \mathbb{N}$, $-\infty < \nu < 1$) *and let* $z_\rho(\tau) := z(b^{1-\rho}\tau^\rho)$, $\tau \in [0,b]$, *where* $\rho \in [1,\infty)$ *if* $\nu \in (0,1)$ *and* $\rho \in \mathbb{N}$ *if* $\nu \le 0$. *Then* $z_\rho \in C^{q,\nu_\rho}(0,b]$ *where* $\nu_\rho := 1 - \rho(1-\nu)$.

For the numerical solution of the boundary value problem (1.1)–(1.2) we use the following method. First we construct an equation of the form (see (2.8))

$$z = Tz \tag{4.2}$$

where $(Tz)(t) := f(t, (Gz)(t) + Q(t))$, $t \in [0,b]$, with G and Q given by (2.6) and (2.7), respectively.

Introducing in (1.5) the change of variables $t = b^{1-\rho}\tau^\rho$, $s = b^{1-\rho}\sigma^\rho$, $\tau,\sigma \in [0,b]$, $\rho \in [1,\infty)$, we obtain that

$$(J^\alpha z)(t) = (J^\alpha_\rho z_\rho)(\tau), \quad t = b^{1-\rho}\tau^\rho,\ \tau \in [0,b],\ \alpha > 0, \tag{4.3}$$

where $z_\rho(\tau) := z(b^{1-\rho}\tau^\rho)$ and

$$(J^\alpha_\rho z_\rho)(\tau) := \frac{\rho b^{(1-\rho)\alpha}}{\Gamma(\alpha)} \int_0^\tau (\tau^\rho - \sigma^\rho)^{\alpha-1}\sigma^{\rho-1} z_\rho(\sigma)\, d\sigma, \quad \tau \in [0,b],\ \alpha > 0. \tag{4.4}$$

Using in (4.2) the change of variables (4.1), we get for $z_\rho(\tau) = z(b^{1-\rho}\tau^\rho)$ an integral equation of the form

$$z_\rho = T_\rho z_\rho \tag{4.5}$$

where

$$(T_\rho z_\rho)(\tau) := f(b^{1-\rho}\tau^\rho, (G_\rho z_\rho)(\tau) + Q_\rho(\tau)), \tag{4.6}$$

$$(G_\rho z_\rho)(\tau) := (J^\alpha_\rho z_\rho)(\tau) - \sum_{\lambda=0}^{n-1} b^{(1-\rho)\lambda}\tau^{\rho\lambda} \sum_{k=1}^{l}\sum_{j=0}^{n_1} \kappa_{\lambda jk} (J^{\alpha-j}_\rho z_\rho)(b_{k\rho}), \tag{4.7}$$

$$Q_\rho(\tau) := Q(b^{1-\rho}\tau^\rho), \qquad b_{k\rho} := b^{(\rho-1)/\rho} b_k^{1/\rho} \in (0,b], \quad \tau \in [0,b],$$

with $\kappa_{\lambda jk}$, Q and J^α_ρ given by (2.4), (2.7) and (4.4), respectively.

Approximations $z_{\rho,N} \in S^{(-1)}_{m-1}(\Pi_N)$ ($m, N \in \mathbb{N}$) to the solution z^*_ρ of Eq. (4.5) we find by collocation method from the conditions

$$z_{\rho,N}(t_{jk}) = (T_\rho z_{\rho,N})(t_{jk}), \quad k = 1,\dots,m,\ j = 1,\dots,N, \tag{4.8}$$

with $\{t_{jk}\}$, defined by (3.2). If $\eta_1 = 0$, then by $z_{\rho,N}(t_{j1})$ we denote the right limit $\lim_{t\to t_{j-1}, t>t_{j-1}} z_{\rho,N}(t)$. If $\eta_m = 1$, then $z_{\rho,N}(t_{jm})$ denotes the left limit $\lim_{t\to t_j, t<t_j} z_{\rho,N}(t)$.

Let $y^*(t)$, $t \in [0,b]$, be the solution of problem (1.1)–(1.2). Using in (2.5) the change of variables (4.1) we see that $y^*_\rho(\tau) := y^*(b^{1-\rho}\tau^\rho)$ can be expressed in the form

$$y^*_\rho = G_\rho z^*_\rho + Q_\rho \tag{4.9}$$

where G_ρ is defined by (4.7) and $Q_\rho(\tau) = Q(b^{1-\rho}\tau^\rho)$, $\tau \in [0,b]$.

Further, approximations $y_{\rho,N}$ to y^*_ρ we find by the formula

$$y_{\rho,N} = G_\rho z_{\rho,N} + Q_\rho \tag{4.10}$$

where $z_{\rho,N} \in S^{(-1)}_{m-1}(\Pi_N)$ is determined by (4.8). Finally, approximations $y_N(t)$ to the solution $y^*(t)$ of problem (1.1)–(1.2) we find by setting

$$y_N(t) := y_{\rho,N}(b^{(\rho-1)/\rho}t^{1/\rho}), \quad t \in [0, b].$$

Note that the conditions (4.8) for finding $z_{\rho,N} \in S^{(-1)}_{m-1}(\Pi_N)$ have an operator equation representation

$$z_{\rho,N} = \mathcal{P}_N T_\rho z_{\rho,N} \tag{4.11}$$

where $\mathcal{P}_N$ and T_ρ are defined by (3.4) and (4.6), respectively.

The collocation conditions (4.8) form a system of equations whose exact form is determined by the choice of a basis in the space $S^{(-1)}_{m-1}(\Pi_N)$. If $\eta_1 > 0$ or $\eta_m < 1$, then we can use the Lagrange fundamental polynomial representation:

$$z_{\rho,N}(\tau) = \sum_{\beta=1}^{N}\sum_{\gamma=1}^{m} c_{\beta\gamma}L_{\beta\gamma}(\tau), \quad \tau \in [0, b], \tag{4.12}$$

where $L_{\beta\gamma}(\tau) = 0$ for $\tau \notin [t_{\beta-1}, t_\beta]$ and

$$L_{\beta\gamma}(\tau) = \prod_{i=1, i\neq\gamma}^{m} \frac{\tau - t_{\beta i}}{t_{\beta\gamma} - t_{\beta i}} \quad \text{for } \tau \in [t_{\beta-1}, t_\beta],\ \gamma = 1, \dots, m,\ \beta = 1, \dots, N.$$

Then $z_{\rho,N} \in S^{(-1)}_{m-1}(\Pi_N)$ and $z_{\rho,N}(t_{jk}) = c_{jk},\ k = 1, \dots, m,\ j = 1, \dots, N$. Searching the solution to (4.8) in the form (4.12), we obtain a system of nonlinear algebraic equations with respect to the coefficients $\{c_{jk}\}$:

$$c_{jk} = f\left(b^{1-\rho}t^\rho_{jk}, \sum_{\beta=1}^{N}\sum_{\gamma=1}^{m}(G_\rho L_{\beta\gamma})(t_{jk})c_{\beta\gamma} + Q_\rho(t_{jk})\right), \quad k = 1, \dots, m,\ j = 1, \dots, N. \tag{4.13}$$

Remark 4.1. In the case of an initial value problem (1.4) we have $G_\rho = J^\alpha_\rho$ (see Remark 2.1). Since $(J^\alpha_\rho L_{\beta\gamma})(t_{jk}) = 0$ if $\beta > j$, in case of problem (1.4) the coefficients $c_{j1}, \dots, c_{jm}$ can be found for every $j = 1, \dots, N$ from the system

$$c_{jk} = f\left(b^{1-\rho}t^\rho_{jk}, \sum_{\beta=1}^{j}\sum_{\gamma=1}^{m}(J^\alpha_\rho L_{\beta\gamma})(t_{jk})c_{\beta\gamma} + Q_\rho(t_{jk})\right), \quad k = 1, \dots, m, \tag{4.14}$$

with m unknowns $c_{jk},\ k = 1, \dots, m$.

Further, having $\{c_{jk}\}$ in hand, we get with the help of (4.10) and (4.12) that

$$y_{\rho,N}(\tau) = \sum_{\beta=1}^{N}\sum_{\gamma=1}^{m} c_{\beta\gamma}(G_\rho L_{\beta\gamma})(\tau) + Q_\rho(\tau), \quad \tau \in [0, b]. \tag{4.15}$$

Note that this algorithm can also be used in the case if in (3.3) $\eta_1 = 0$ and $\eta_m = 1$. In this case we have $t_{jm} = t_{j+1,1} = t_j$ and $c_{jm} = c_{j+1,1}$ $(j = 1, \dots, N-1)$, and hence in the system (4.12) there are $(m-1)N + 1$ equations and unknowns.

5. Convergence analysis

First of all we introduce some notations and definitions. Let X be a Banach space with a norm $\|x\|,\ x \in X$. A sequence $\{A_N\}_{N\in\mathbb{N}}$ of operators $A_N \in \mathcal{L}(X, X)$ is called compactly converging to $A \in \mathcal{L}(X; X)$ (we write $A_N \to A$ compactly) if $A_N x \to Ax$ as $N \to \infty$ for every $x \in X$ and for any bounded sequence $\{x_N\}_{N\in\mathbb{N}},\ x_N \in X$, it follows that the sequence $\{A_N x_N\}_{N\in\mathbb{N}}$ is relatively compact in X (i.e. every subsequence $\{A_N x_N\}_{N\in\mathbb{N}'\subset\mathbb{N}}$ contains a subsequence $\{A_N x_N\}_{N\in\mathbb{N}''\subset\mathbb{N}'}$ converging in X).

Let us consider the nonlinear equations

$$x = Sx \tag{5.1}$$

and

$$x = S_N x, \quad N \in \mathbb{N}, \tag{5.2}$$

where $S : \mathcal{B} \to X$ and $S_N : \mathcal{B} \to X$ are nonlinear operators defined on an open set $\mathcal{B} \subset X$.

We recall that $S:\mathcal{B}\to X$ is called Fréchet differentiable at $x^0\in\mathcal{B}$ if there exists a linear operator $S'(x^0)\in\mathcal{L}(X,X)$ such that

$$\|Sx-Sx^0-S'(x^0)(x-x^0)\|/\|x-x^0\|\to 0 \quad \text{as } \|x-x^0\|\to 0;$$

in this case $S'(x^0)$ is the (unique) Fréchet derivative of S at x^0.

To study the convergence behavior of our method we use the following result from the approximation theory by Vainikko (see Theorem 2 in [36]).

Lemma 5.1. *Let the following conditions be fulfilled:*

1^0 *Eq. (5.1) has a solution $x^*\in\mathcal{B}$, and the operator S is Fréchet differentiable at x^*;*
2^0 *there is a positive number δ such that the operators S_N $(N\in\mathbb{N})$ are Fréchet differentiable in the ball $\|x-x^*\|\le\delta$ which is assumed to be contained in $\mathcal{B}$, and for any $\varepsilon>0$ there is a $\delta_\varepsilon\in(0,\delta]$ such that for every $N\in\mathbb{N}$*

$$\|S_N'(x)-S_N'(x^*)\|_{\mathcal{L}(X,X)}\le\varepsilon \quad \text{whenever } \|x-x^*\|\le\delta_\varepsilon;$$

3^0 *$\|S_Nx^*-Sx^*\|\to 0$ as $N\to\infty$;*
4^0 *$S_N'(x^*)\to S'(x^*)$ compactly, whereby $S_N'(x^*)\in\mathcal{L}(X,X)$ $(N\in\mathbb{N})$ are compact and the homogeneous equation $x=S'(x^*)x$ has in X only the trivial solution $x=0$.*

Then there exist $N_0\in\mathbb{N}$ and $\delta_0\in(0,\delta]$ such that Eq. (5.2) has for $N\ge N_0$ a unique solution x_N in the ball $\|x-x^*\|\le\delta_0$. Thereby $x_N\to x^*$ as $N\to\infty$ and the following error estimate holds:

$$\|x_N-x^*\|\le c\,\|S_Nx^*-Sx^*\|, \quad N\ge N_0. \tag{5.3}$$

Here c is a positive constant not depending on N.

The following lemma presents some properties of J_ρ^α which follow from (4.3) and from the corresponding properties of the Riemann–Liouville integral operator J^α defined by (1.5).

Lemma 5.2. *Let $\alpha>0$ and $\rho\ge 1$ be some given real numbers. Then J_ρ^α defined by (4.4) is linear, bounded and compact as an operator from $L^\infty(0,b)$ into $C[0,b]$. Moreover, we have for any $x\in L^\infty(0,b)$ that*

$$J_\rho^\alpha J_\rho^\beta x=J_\rho^{\alpha+\beta}x, \quad \alpha>0,\ \beta>0,\ \rho\ge 1.$$

Now we are ready to prove the convergence of method (4.10)–(4.11) and study its attainable order of convergence for arbitrary collocation parameters $\eta_1,\dots,\eta_m$ satisfying (3.3).

Theorem 5.1. *Suppose (1.3) and let problem (1.1)–(1.2) have a solution $y^*\in C[0,b]$ such that $z^*:=D_*^\alpha y^*\in C[0,b]$. Let $f:\bar\Omega\to\mathbb{R}$ be a continuous function such that $\frac{\partial}{\partial y}f(t,y)$ is continuous for $(t,y)\in\bar\Omega$, $\frac{\partial^2}{\partial y^2}f(t,y)$ is continuous for $(t,y)\in\Omega$ and*

$$\left|\frac{\partial^j}{\partial y^j}f(t,y)\right|\le\psi(|y|), \quad (t,y)\in\Omega,\ j=0,\ 1,\ 2.$$

Here Ω and $\bar\Omega$ are defined by (2.10) and $\psi:[0,\infty)\to\mathbb{R}$ is a monotonically increasing function. Moreover, assume that from all polynomials y of degree $n-1$ $(n=\lceil\alpha\rceil)$ only $y=0$ satisfies the homogeneous boundary conditions (2.3) and from all solutions $y\in C[0,b]$ of the linear homogeneous (fractional) differential equation

$$(D_*^\alpha y)(t)=\frac{\partial}{\partial y}f\big(t,y^*(t)\big)y(t), \quad t\in[0,b], \tag{5.4}$$

only $y=0$ satisfies the conditions (2.3). Finally, let $m\in\mathbb{N}$ and assume that the collocation points (3.2) with grid points (3.1) and arbitrary parameters $\eta_1,\dots,\eta_m$ satisfying (3.3) are used.

Then there exist $N_0\in\mathbb{N}$ and $\delta_0>0$ such that, for all $N\ge N_0$, Eq. (4.11) possesses a unique solution $z_{\rho,N}\in S_{m-1}^{(-1)}(\Pi_N)$ in the ball $\|x-z_\rho^*\|_\infty\le\delta_0$ where $z_\rho^*(\tau)=z^*(b^{1-\rho}\tau)$, $\tau\in[0,b]$, $z^*=D_*^\alpha y^*$ and $\rho\ge 1$. Moreover,

$$\|y_N-y^*\|_\infty\to 0 \quad \text{as } N\to\infty, \tag{5.5}$$

where $y_N(t)=y_{\rho,N}(b^{(\rho-1)/\rho}t^{1/\rho})$, $t\in[0,b]$, and $y_{\rho,N}$ is defined with the help of $z_{\rho,N}$ by (4.10).

If, in addition, the assumptions of Theorem 2.1 are fulfilled with $q=m$ and some $\nu\in[1-\alpha,1)$, then for all $N\ge N_0$ and $r\ge 1$ the following error estimate holds:

$$\|y_N - y^*\|_\infty \le c \begin{cases} N^{-\rho r(1-\nu)} & \text{for} \quad 1 \le \rho r < \dfrac{m}{1-\nu} \\ N^{-m}(1+\log N) & \text{for} \quad \rho r = \dfrac{m}{1-\nu} = 1 \\ N^{-m} & \text{for} \quad \rho r = \dfrac{m}{1-\nu} > 1 \quad \text{or} \quad \rho r > \dfrac{m}{1-\nu} \end{cases}. \tag{5.6}$$

Here c is a constant not depending on N, $\rho \in [1, \infty)$ if $\nu \in (0, 1)$ and $\rho \in \mathbb{N}$ if $\nu \le 0$.

Proof. To apply Lemma 5.1 we consider (4.5) and (4.11) as operator Eqs. (5.1) and (5.2) in the space $X := L^\infty(0, b)$ where $S := T_\rho$ and $S_N := \mathcal{P}_N T_\rho$ with T_ρ and $\mathcal{P}_N$, defined by (4.6) and (3.4), respectively.

First we find the Fréchet derivative $T'_\rho(x^0)$ for T_ρ at $x^0 \in L^\infty(0, b)$. If x and x^0 belong to $L^\infty(0, b)$, then, due to Taylor formula,

$$\begin{aligned}(T_\rho x)(\tau) - (T_\rho x^0)(\tau) &= \frac{\partial}{\partial y} f\big(t, (G_\rho x^0)(\tau) + Q_\rho(\tau)\big)\big(G_\rho(x - x^0)\big)(\tau) \\ &\quad + \frac{1}{2}\frac{\partial^2}{\partial y^2} f\big(t, (G_\rho x^0)(\tau) + \xi(\tau)(G_\rho(x - x^0))(\tau) + Q_\rho(\tau)\big)\big[\big(G_\rho(x - x^0)\big)(\tau)\big]^2, \end{aligned} \tag{5.7}$$

where $\xi(\tau) \in [0, 1]$ and $t = b^{1-\rho}\tau \in (0, b]$. From this it follows that

$$\big(T'_\rho(x^0)x\big)(\tau) = \left[\frac{\partial}{\partial y} f(b^{1-\rho}\tau, y)\bigg|_{y=(G_\rho x^0)(\tau)+Q_\rho(\tau)}\right](G_\rho x)(\tau), \tag{5.8}$$

where $\tau \in [0, b]$ and $x^0, x \in L^\infty(0, b)$. Observe that $T'_\rho(x^0) \in \mathcal{L}(L^\infty(0, b), C[0, b])$ if $x^0 \in L^\infty(0, b)$. From (5.7) we see that

$$S'_N(x^0)x = (\mathcal{P}_N T_\rho)'(x^0)z = \mathcal{P}_N\big(T'_\rho(x^0)x\big), \quad x^0,\ x \in L^\infty(0, b),$$

with $T'_\rho(x^0)x$, defined by (5.8).

It follows from Lemma 5.2 that G_ρ (defined by (4.7)) is linear, bounded and compact as an operator from $L^\infty(0, b)$ into $C[0, b]$. Since from all solutions y of Eq. (5.4) only $y = 0$ satisfies (2.3), Eq. $x = T'_\rho(z^*_\rho)x$ has in $L^\infty(0, b)$ only the trivial solution $x = 0$. Using these observations and Lemma 5.2 we can check that the operators $S = T_\rho$ and $S_N = \mathcal{P}_N T_\rho$ satisfy the conditions $1^0 - 4^0$ of Lemma 5.1. Hence, in agreement with Lemma 5.1, there exist $N_0 \in \mathbb{N}$ and $\delta_0 > 0$ such that, for all $N \ge N_0$, Eq. (4.11) possesses a unique solution $z_{\rho,N}$ in the ball $\|x - z^*_\rho\|_\infty \le \delta_0$, and

$$\|z_{\rho,N} - z^*_\rho\|_\infty \le c\,\|\mathcal{P}_N z^*_\rho - z^*_\rho\|_\infty,$$

with a positive constant c which is independent of N. Since $y_{\rho,N} = G_\rho z_{\rho,N} + Q_\rho$ and $y^*_\rho = G_\rho z^*_\rho + Q_\rho$, we obtain that

$$\|y_N - y^*\|_\infty \le c\,\|z_{\rho,N} - z^*_\rho\|_\infty \le c_1\,\|\mathcal{P}_N z^*_\rho - z^*_\rho\|_\infty, \tag{5.9}$$

with some positive constants c and c_1 which are independent of N. This together with $z^*_\rho \in C[0, b]$ and (3.5) yields the convergence (5.5).

If the assumptions of Theorem 2.1 are fulfilled with $q = m$ and $\nu \in [1 - \alpha, 1)$, then $z^* \in C^{m,\nu}(0, b]$, and we get by Lemma 4.1 that $z^*_\rho \in C^{m,\nu_\rho}(0, b]$, $\nu_\rho = 1 - \rho(1 - \nu)$. This together with (5.9) and (3.6) yields the estimate (5.6). □

6. Superconvergence results

It follows from Theorem 5.1 that in the case of sufficiently smooth $f(t, y)$, using sufficiently large values of ρr, for method (4.10)–(4.11) by every choice of collocation parameters $0 \le \eta_1 < \cdots < \eta_m \le 1$ a convergence of order $O(N^{-m})$ can be expected. From Theorem 6.1 we see that by a careful choice of parameters $\eta_1, \ldots, \eta_m$ it is possible to establish a faster convergence of our method. In order to study the superconvergence properties of our method by a special choice of collocation parameters we need the following lemma from [35].

Lemma 6.1. *Let $z \in C^{m+1,\nu}(0, b]$ with some $m \in \mathbb{N}$ and $\nu \in (-\infty, 1)$. Let $N \in \mathbb{N}$, $\alpha \in (0, 1]$, $r \in [1, \infty)$, $\rho \in [1, \infty)$ if $\nu \in (0, 1)$ and $\rho \in \mathbb{N}$ if $\nu \le 0$, $z_\rho(\tau) = z(b^{1-\rho}\tau^\rho)$, $\tau \in [0, b]$. Let $\{t_j\}$, $\mathcal{P}_N$ and J^α_ρ be defined by (3.1), (3.4) and (4.4), respectively. Moreover, assume that a quadrature approximation*

$$\int_0^1 g(x)\,dx \approx \sum_{k=1}^m w_k g(\eta_k) \tag{6.1}$$

with the knots $\{\eta_k\}$ satisfying (3.3) and appropriate weights $\{w_k\}$ is exact for all polynomials g of degree m. Then the following estimate holds:

$$\|J^{\alpha}_{\rho}(\mathcal{P}_N z_\rho - z_\rho)\|_\infty \le c \begin{Bmatrix} E_N(m,\alpha,\nu,\rho r) & \text{if} & 0<\alpha<1 \\ E^*_N(m,\nu,\rho r) & \text{if} & \alpha = 1 \end{Bmatrix}. \tag{6.2}$$

Here c is a constant not depending on N, and

$$E_N(m,\alpha,\nu,\rho r) := \begin{Bmatrix} N^{-\rho r(\alpha+1-\nu)} & \text{if} & 1 \le \rho r < \dfrac{m+\alpha}{\alpha+1-\nu} \\ N^{-m-\alpha}(1+\log N) & \text{if} & \rho r = \dfrac{m+\alpha}{\alpha+1-\nu} = 1 \\ N^{-m-\alpha} & \text{if} & \rho r = \dfrac{m+\alpha}{\alpha+1-\nu} > 1 \\ & \text{or} & \rho r > \dfrac{m+\alpha}{\alpha+1-\nu} \end{Bmatrix}, \tag{6.3}$$

$$E^*_N(m,\nu,\rho r) := \begin{Bmatrix} N^{-\rho r(2-\nu)} & \text{if} & 1 \le \rho r < \dfrac{m+1}{2-\nu} \\ N^{-m-1}(1+\log N)^2 & \text{if} & \rho r = \dfrac{m+1}{2-\nu} = 1 \\ N^{-m-1}(1+\log N) & \text{if} & \rho r = \dfrac{m+1}{2-\nu} > 1 \\ N^{-m-1} & \text{if} & \rho r > \dfrac{m+1}{2-\nu} \end{Bmatrix}. \tag{6.4}$$

Theorem 6.1. *Assume that the following conditions are fulfilled:*

(i) *problem (1.1)–(1.2) has a solution $y^* \in C[0,b]$ and (1.3) holds;*
(ii) *$m \in \mathbb{N}$ and problem (1.1)–(1.2) satisfies the conditions of Theorem 2.1 with $q = m+1$ and $\nu \in [1-\alpha, 1)$;*
(iii) *from all solutions $y \in C[0,b]$ to Eq. (5.4) only $y = 0$ satisfies the boundary conditions (2.3);*
(iv) *ρ is a smoothing parameter (see (4.1)) such that $\rho \in [1,\infty)$ if $\nu \in (0,\ 1)$ and $\rho \in \mathbb{N}$ if $\nu \le 0$;*
(v) *$\mathcal{P}_N$ is defined by (3.4) where the collocation points (3.2) are generated by grid points (3.1) and by parameters $0 \le \eta_1 < \eta_2 < \cdots < \eta_m \le 1$ so that a quadrature approximation (6.1) is exact for all polynomials of degree m.*

Then there exist $N_0 \in \mathbb{N}$ and $\delta_0 > 0$ such that for all $N \ge N_0$ Eq. (4.11) possesses a unique solution $z_{\rho,N}$ in the ball $\|x - z^*_\rho\| \le \delta_0$ where $z^*_\rho(\tau) = z^*(b^{1-\rho}\tau),\ \tau \in [0,b],\ z^* = D^\alpha_* y^*$. Moreover, for $N \ge N_0$ the following error estimates hold:

$$\|y_N - y^*\|_\infty \le c \begin{Bmatrix} N^{-\rho r(\alpha-n_1+1-\nu)} & \text{if} & 1 \le \rho r < \dfrac{m+\alpha-n_1}{\alpha-n_1+1-\nu} \\ N^{-m-\alpha+n_1}(1+\log N) & \text{if} & \rho r = \dfrac{m+\alpha-n_1}{\alpha-n_1+1-\nu} = 1 \\ N^{-m-\alpha+n_1} & \text{if} & \rho r = \dfrac{m+\alpha-n_1}{\alpha-n_1+1-\nu} > 1 \\ & \text{or} & \rho r > \dfrac{m+\alpha-n_1}{\alpha-n_1+1-\nu} \end{Bmatrix} \tag{6.5}$$

for $\alpha - n_1 < 1$ and

$$\|y_N - y^*\|_\infty \le c \begin{Bmatrix} N^{-\rho r(2-\nu)} & \text{if} & 1 \le \rho r < \dfrac{m+1}{2-\nu} \\ N^{-m-1}(1+\log N)^2 & \text{if} & \rho r = \dfrac{m+1}{2-\nu} = 1 \\ N^{-m-1}(1+\log N) & \text{if} & \rho r = \dfrac{m+1}{2-\nu} > 1 \\ N^{-m-1} & \text{if} & \rho r > \dfrac{m+1}{2-\nu} \end{Bmatrix} \tag{6.6}$$

for $\alpha - n_1 \ge 1$. In (6.5) and (6.6) *$y_N(t) = y_{\rho,N}(b^{(\rho-1)/\rho} t^{1/\rho}),\ t \in [0,b]$, with $y_{\rho,N}$ given by (4.10), $r \in [1,\infty)$ is given by (3.1) and c is a positive constant not depending on N.*

Proof. Since $q = m+1 \geq 2$, it follows from Theorem 5.1 that there exist $\hat{N}_0 \in \mathbb{N}$ and $\hat{\delta}_0 > 0$ such that for $N \geq \hat{N}_0$ Eq. (4.11) has a unique solution $z_{\rho,N}$ in the ball $\|x - z^*_\rho\|_\infty \leq \hat{\delta}_0$. Denote

$$\hat{z}_{\rho,N} := T_\rho z_{\rho,N}, \quad N \geq \hat{N}_0, \tag{6.7}$$

with T_ρ defined by the formula (4.6). Taking into account (4.11), we obtain that $\mathcal{P}_N \hat{z}_{\rho,N} = z_{\rho,N}$. Substituting $z_{\rho,N} = \mathcal{P}_{\rho,N}\hat{z}_{\rho,N}$ into (6.7) we see that $\hat{z}_{\rho,N}$ is a solution of the equation

$$\hat{z}_{\rho,N} = T_\rho \mathcal{P}_N \hat{z}_{\rho,N}, \quad N \geq \hat{N}_0. \tag{6.8}$$

Consider Eqs. (4.5) and (6.8) as operator Eqs. (5.1) and (5.2) in the space $X := C[0,b]$ where $S := T_\rho$ and $S_N := T_\rho \mathcal{P}_N$. In a similar way as we obtained the formula (5.8), we get for the Fréchet derivative of $S_N = T_\rho \mathcal{P}_N$ at $x^0 \in C[0,b]$ the following formula:

$$\big(S'_N(x^0)x\big)(\tau) = \left[\frac{\partial}{\partial y} f(b^{1-\rho}\tau, y)\Big|_{y=(G_\rho \mathcal{P}_N x^0)(\tau)+Q_\rho(\tau)}\right](G_\rho \mathcal{P}_N x)(\tau), \quad \tau \in [0,b],\ x \in C[0,b].$$

Moreover, we can check that the operators $S = T_\rho$ and $S_N = T_\rho \mathcal{P}_N$ satisfy the conditions $1^0 - 4^0$ of Lemma 5.1. From this lemma it follows that there exist $N_0 \geq \hat{N}_0$ and $\delta_0 > 0$ such that, for $N \geq N_0$, Eq. (6.8) possesses a unique solution $\hat{z}_{\rho,N}$ in the ball $\|x - z^*_\rho\|_\infty \leq \delta_0$ whereby

$$\|\hat{z}_{\rho,N} - z^*_\rho\|_\infty \leq c\, \|T_\rho \mathcal{P}_N z^*_\rho - T_\rho z^*_\rho\|_\infty, \quad N \geq N_0. \tag{6.9}$$

Here and below by c and also c_1 we denote positive constants which are independent of N. As

$$(T_\rho \mathcal{P}_N z^*_\rho)(\tau) - (T_\rho z^*_\rho)(\tau) = f\big(b^{1-\rho}\tau, (G_\rho \mathcal{P}_N z^*_\rho)(\tau) + Q_\rho(\tau)\big) - f\big(b^{1-\rho}\tau, (G_\rho z^*_\rho)(\tau) + Q_\rho(\tau)\big), \quad \tau \in [0,b],$$

then with the help of Lagrange formula we obtain that

$$\|T_\rho \mathcal{P}_N z^*_\rho - T_\rho z^*_\rho\|_\infty \leq c\, \|G_\rho(\mathcal{P}_N z^*_\rho - z^*_\rho)\|_\infty. \tag{6.10}$$

Using (4.7) and Lemma 5.2 we get

$$\begin{aligned} \|G_\rho(\mathcal{P}_N z^*_\rho - z^*_\rho)\|_\infty &\leq \|J^\alpha_\rho(\mathcal{P}_N z^*_\rho - z^*_\rho)\|_\infty \\ &+ c\sum_{k=1}^{l}\sum_{j=0}^{n_1}\big|\big(J^{\alpha-j}_\rho(\mathcal{P}_N z^*_\rho - z^*_\rho)\big)(b_{k\rho})\big| \leq c_1\, \|J^{\alpha-n_1}_\rho(\mathcal{P}_N z^*_\rho - z^*_\rho)\|_\infty \end{aligned} \tag{6.11}$$

where $b_{k\rho} = b^{(\rho-1)/\rho} b_k^{1/\rho} \in [0,b]$. From (6.9)–(6.11) it follows that

$$\|\hat{z}_{\rho,N} - z^*_\rho\|_\infty \leq c\, \|J^{\alpha-n_1}_\rho(\mathcal{P}_N z^*_\rho - z^*_\rho)\|_\infty, \quad N \geq N_0. \tag{6.12}$$

Further, as $y_{\rho,N} = G_\rho z_{\rho,N} + Q_\rho$, $y^*_\rho = G_\rho z^*_\rho + Q_\rho$ and $z_{\rho,N} = \mathcal{P}_N \hat{z}_{\rho,N}$, then

$$\begin{aligned} \|y_N - y^*\|_\infty &= \|y_{\rho,N} - y^*_\rho\|_\infty = \|G_\rho(z_{\rho,N} - z^*_\rho)\|_\infty \\ &\leq \|G_\rho \mathcal{P}_N(\hat{z}_{\rho,N} - z^*_\rho)\|_\infty + \|G_\rho(\mathcal{P}_N z^*_\rho - z^*_\rho)\|_\infty, \quad N \geq N_0. \end{aligned}$$

This together with (6.11) and (6.12) yields

$$\|y_N - y^*\|_\infty \leq c\, \|J^{\alpha-n_1}_\rho(\mathcal{P}_N z^*_\rho - z^*_\rho)\|_\infty, \quad N \geq N_0. \tag{6.13}$$

Because of Theorem 2.1 we have $z^* \in C^{m+1,\nu}(0,b]$ and, due to Lemma 5.2,

$$\|J^{\alpha-n_1}_\rho(\mathcal{P}_N z^*_\rho - z^*_\rho)\|_\infty \leq c\, \|J^1_\rho(\mathcal{P}_N z^*_\rho - z^*_\rho)\|_\infty, \quad \alpha - n_1 \geq 1, \quad N \geq N_0.$$

This together with (6.13) and Lemma 6.1 yields the estimates (6.5) and (6.6). □

7. Numerical experiments

In this section, we present some numerical experiments to demonstrate the accuracy of the modified spline collocation method (4.10)–(4.11) and compare the actual convergence rate with the theoretical error estimates of Theorem 6.1. For obtaining numerical results Fortran-98 language has been used.

Example 1. Consider the following initial value problem:

$$\left(D_*^{1/3}y\right)(t) = f(t, y(t)), \quad y(0) = 1,\ 0 \le t \le 1, \tag{7.1}$$

where

$$f(t, y) := \frac{1}{10}ty^3 - \frac{1}{10}t(t^{2/3}+1)^3 + \frac{\Gamma(5/3)}{\Gamma(4/3)}t^{1/3}.$$

The function $f(t, y)$ satisfies the conditions (2.11) and (2.12) with $\nu = 2/3$ and arbitrary $q \in \mathbb{N}$. An exact solution of this problem is $y^*(t) = t^{2/3} + 1$ (see Fig. 1(a)). Its Caputo derivative is $z^*(t) := \left(D_*^{1/3}y^*\right)(t) = \frac{\Gamma(5/3)}{\Gamma(4/3)}t^{1/3}$.

To solve (7.1) with method (4.10)–(4.11) we set $z := D_*^{1/3}y$. Then a solution of (7.1) can be presented in the form $y^*(t) = (J^{1/3}z^*)(t) + 1$ where z^* is a solution of the integral equation

$$z(t) = f\left(t, (J^{1/3}z)(t) + 1\right), \quad t \in [0,\ 1].$$

Using a change of variables $t = \tau^\rho,\ \tau \in [0,\ 1],\ \rho \ge 1$, and notations

$$y_\rho(\tau) := y(\tau^\rho),\ z_\rho(\tau) := z(\tau^\rho), \quad \tau \in [0,\ 1],\ \rho \ge 1, \tag{7.2}$$

we get that

$$y^*_\rho(\tau) = (J_\rho^{1/3}z^*_\rho)(\tau) + 1, \quad \tau \in [0,\ 1],$$

where $z^*_\rho = z^*(\tau^\rho)$ is a solution of the integral equation

$$z_\rho(\tau) = f\left(\tau^\rho, (J_\rho^{1/3}z_\rho)(\tau) + 1\right), \quad \tau \in [0,\ 1]. \tag{7.3}$$

Here

$$(J_\rho^{1/3}z_\rho)(\tau) = \frac{\rho}{\Gamma(1/3)}\int_0^\tau (\tau^\rho - \sigma^\rho)^{-2/3}\sigma^{\rho-1}z_\rho(\sigma)\,d\sigma, \quad \tau \in [0,\ 1].$$

Since $\tau = t^{1/\rho}$, we have $y^*(t) = y^*_\rho(t^{1/\rho}),\ t \in [0,\ 1]$.

For the numerical solution of Eq. (7.3) we use spline collocation method (4.8). We determine approximations $z_{\rho,N}$ to the solution z^*_ρ of (7.3) in the form (4.12), finding the coefficients $\{c_{jk}\}$ from collocation conditions (4.14):

$$c_{jk} = f\left(t_{jk}^\rho, \sum_{\beta=1}^{j}\sum_{\gamma=1}^{m}(J_\rho^{1/3}L_{\beta\gamma})(t_{jk})c_{\beta\gamma} + 1\right), \quad k = 1, \ldots, m,\ j = 1, \ldots, N. \tag{7.4}$$

For calculations we use grid points $t_j = (j/N)^r,\ j = 0, \ldots, N,\ r \ge 1$, and collocation points $t_{jk} = t_{j-1} + \eta_k(t_j - t_{j-1}),\ k = 1, \ldots, m,\ j = 1, \ldots, N$, where in the case $m = 2,\ \eta_1 = (3 - \sqrt{3})/6,\ \eta_2 = 1 - \eta_1$, and in the case $m = 3,\ \eta_1 = (5 - \sqrt{15})/10,\ \eta_2 = 1/2,\ \eta_3 = 1 - \eta_1$. These parameters $\{\eta_k\}$ are the node points of the Gaussian quadrature rule (6.1) which in the case $m = 2$ is exact for all polynomials of degree 3 and in the case $m = 3$ is exact for all polynomials of degree 5.

Weakly singular integrals in (7.4) are found by the package QUADPACK and the coefficients $\{c_{jk}\}$ in (4.12) are determined from the nonlinear system (7.4) by Newton method. Using these coefficients we find approximations $y_{\rho,N}$ to the solution y^* of problem (7.1) by the formula (4.15):

$$y_{\rho,N}(\tau) = \sum_{\beta=1}^{N}\sum_{\gamma=1}^{m}c_{\beta\gamma}(J_\rho^{1/3}L_{\beta\gamma})(\tau) + 1, \quad \tau \in [0,\ 1].$$

In Tables 7.1–7.3 some results of numerical experiments for different values of the parameters N, ρ and r are presented. The results in Tables 7.1 and 7.2 are calculated using uniform grids ($r = 1$) and the results in Table 7.3 using nonuniform grids ($r > 1$). The quantities ε_N in these tables are the approximate values of the norm $\|y_N - y^*\|_\infty$, calculated as follows:

$$\varepsilon_N := \max_{j=1,\ldots,N}\ \max_{k=0,\ldots,10} |y_N(s_{jk}) - y^*(s_{jk})|. \tag{7.5}$$

Here $s_{jk} := t_{j-1} + k(t_j - t_{j-1})/10,\ k = 0, \ldots, 10,\ j = 1, \ldots, N$, and $y_N(t) = y_{\rho,N}(t^{1/\rho}),\ 0 \le t \le 1$. The ratios $\varrho_N := \varepsilon_{N/2}/\varepsilon_N$, characterizing the observed convergence rate, are also presented. Moreover, in Fig. 1(b) the graph of the error $y_N(t) - y^*(t)$ for the problem (7.1) in the case $m = 3, r = 2, \rho = 2, N = 64$ is displayed. We see that the approximate solution $y_N(t)$ oscillates around the exact solution $y^*(t)$ for $t \in [0, 1]$.

Since for problem (7.1) $\alpha = 1/3,\ n_1 = 0$ and $\nu = 2/3$, it follows from (6.5) that, for $\rho \ge 1,\ r \ge 1$ and sufficiently large N, in case $m = 2$,

$$\varepsilon_N \le c\begin{Bmatrix} N^{-2\rho r/3} & \text{if} & 1 \le \rho r \le 3.5 \\ N^{-7/3} & \text{if} & \rho r \ge 3.5 \end{Bmatrix},$$

Table 7.1
Numerical results for problem (7.1) in the case $m = 2$.

N	$r=1,\ \rho=1$		$r=1,\ \rho=2$		$r=1,\ \rho=3.5$		$r=1,\ \rho=4$	
	ε_N	ϱ_N	ε_N	ϱ_N	ε_N	ϱ_N	ε_N	ϱ_N
4	$2.27 \cdot 10^{-2}$		$2.80 \cdot 10^{-3}$		$6.10 \cdot 10^{-4}$		$9.17 \cdot 10^{-4}$	
8	$1.42 \cdot 10^{-2}$	1.592	$1.12 \cdot 10^{-3}$	2.513	$1.21 \cdot 10^{-4}$	5.039	$1.93 \cdot 10^{-4}$	4.754
16	$8.95 \cdot 10^{-3}$	1.589	$4.43 \cdot 10^{-4}$	2.519	$2.40 \cdot 10^{-5}$	5.040	$3.98 \cdot 10^{-5}$	4.844
32	$5.64 \cdot 10^{-3}$	1.588	$1.76 \cdot 10^{-4}$	2.520	$4.77 \cdot 10^{-6}$	5.040	$7.95 \cdot 10^{-6}$	5.011
64	$3.55 \cdot 10^{-3}$	1.588	$6.98 \cdot 10^{-5}$	2.520	$9.46 \cdot 10^{-7}$	5.040	$1.57 \cdot 10^{-6}$	5.072
128	$2.24 \cdot 10^{-3}$	1.587	$2.77 \cdot 10^{-5}$	2.520	$1.88 \cdot 10^{-7}$	5.040	$3.08 \cdot 10^{-7}$	5.093
256	$1.41 \cdot 10^{-3}$	1.587	$1.10 \cdot 10^{-5}$	2.520	$3.73 \cdot 10^{-8}$	5.040	$6.03 \cdot 10^{-8}$	5.103
		1.587		2.520		5.040		5.040

Table 7.2
Numerical results for problem (7.1) in the case $m = 3$.

N	$r=1,\ \rho=1$		$r=1,\ \rho=2$		$r=1,\ \rho=4$		$r=1,\ \rho=5$	
	ε_N	ϱ_N	ε_N	ϱ_N	ε_N	ϱ_N	ε_N	ϱ_N
4	$7.27 \cdot 10^{-3}$		$7.09 \cdot 10^{-4}$		$8.43 \cdot 10^{-5}$		$3.68 \cdot 10^{-5}$	
8	$4.57 \cdot 10^{-3}$	1.591	$2.81 \cdot 10^{-4}$	2.521	$1.33 \cdot 10^{-5}$	6.349	$3.65 \cdot 10^{-6}$	10.08
16	$2.88 \cdot 10^{-3}$	1.589	$1.12 \cdot 10^{-4}$	2.520	$2.09 \cdot 10^{-6}$	6.350	$3.62 \cdot 10^{-7}$	10.08
32	$1.81 \cdot 10^{-3}$	1.588	$4.43 \cdot 10^{-5}$	2.520	$3.29 \cdot 10^{-7}$	6.350	$3.60 \cdot 10^{-8}$	10.08
64	$1.14 \cdot 10^{-3}$	1.588	$1.76 \cdot 10^{-5}$	2.520	$5.19 \cdot 10^{-8}$	6.350	$3.57 \cdot 10^{-9}$	10.08
128	$7.19 \cdot 10^{-4}$	1.587	$6.98 \cdot 10^{-6}$	2.520	$8.17 \cdot 10^{-9}$	6.350	$4.50 \cdot 10^{-10}$	7.93
256	$4.53 \cdot 10^{-4}$	1.587	$2.77 \cdot 10^{-6}$	2.520	$1.29 \cdot 10^{-9}$	6.350	$7.90 \cdot 10^{-10}$	0.57
		1.587		2.520		6.350		10.08

Table 7.3
Numerical results for problem (7.1) in the case $m = 3$.

N	$r=2,\ \rho=1$		$r=2,\ \rho=2$		$r=2,\ \rho=2.5$		$r=3,\ \rho=2$	
	ε_N	ϱ_N	ε_N	ϱ_N	ε_N	ϱ_N	ε_N	ϱ_N
4	$2.88 \cdot 10^{-3}$		$1.48 \cdot 10^{-4}$		$5.42 \cdot 10^{-5}$		$2.62 \cdot 10^{-4}$	
8	$1.14 \cdot 10^{-3}$	2.521	$2.34 \cdot 10^{-5}$	6.329	$5.69 \cdot 10^{-6}$	9.53	$2.86 \cdot 10^{-5}$	9.16
16	$4.53 \cdot 10^{-4}$	2.520	$3.69 \cdot 10^{-6}$	6.349	$5.74 \cdot 10^{-7}$	9.91	$3.06 \cdot 10^{-6}$	9.34
32	$1.80 \cdot 10^{-4}$	2.520	$5.81 \cdot 10^{-7}$	6.350	$5.73 \cdot 10^{-8}$	10.02	$3.16 \cdot 10^{-7}$	9.68
64	$7.13 \cdot 10^{-5}$	2.520	$9.16 \cdot 10^{-8}$	6.350	$5.70 \cdot 10^{-9}$	10.05	$3.21 \cdot 10^{-8}$	9.84
128	$2.83 \cdot 10^{-5}$	2.520	$1.44 \cdot 10^{-8}$	6.350	$5.92 \cdot 10^{-10}$	9.62	$3.26 \cdot 10^{-9}$	9.83
256	$1.12 \cdot 10^{-5}$	2.520	$2.27 \cdot 10^{-9}$	6.350	$6.07 \cdot 10^{-10}$	0.97	$5.47 \cdot 10^{-10}$	5.97
		2.520		6.350		10.08		10.08

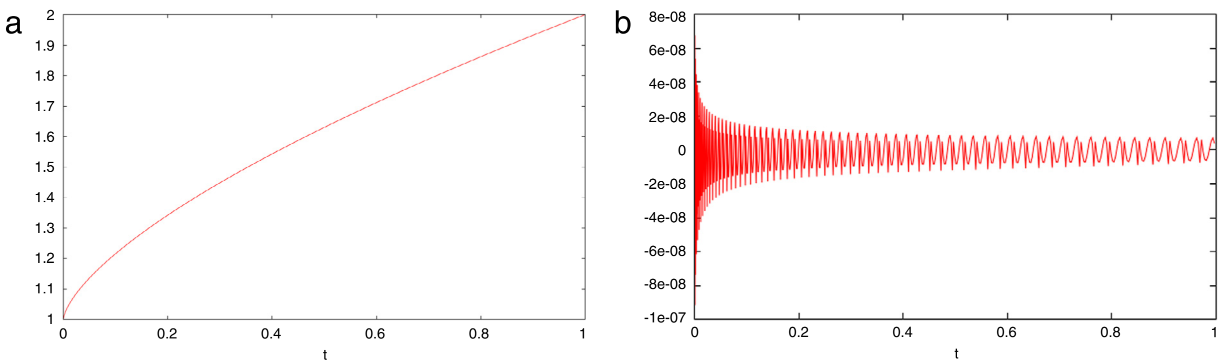

Fig. 1. (a) The exact solution $y^*(t) = t^{\frac{2}{3}} + 1$ for (7.1). (b) The error $y_N(t) - y^*(t)$ for (7.1) in the case $m = 3$, $r = 2$, $\rho = 2$, $N = 64$.

and in case $m = 3$,

$$\varepsilon_N \le c \begin{Bmatrix} N^{-2\rho r/3} & \text{if} \quad 1 \le \rho r \le 5 \\ N^{-10/3} & \text{if} \quad \rho r \ge 5 \end{Bmatrix}.$$

Due to these error estimates the ratios ϱ_N in the case $m = 2$ for $\rho r = 1,\ \rho r = 2$ and $\rho r \ge 3.5$ ought to be approximately $2^{2/3} \approx 1.587,\ 2^{4/3} \approx 2.520$ and $2^{7/3} \approx 5.040$, respectively. In the case $m = 3$ for $\rho r = 1,\ \rho r = 2,\ \rho = 4$ and $\rho r \ge 5$

the ratios ϱ_N ought to be approximately $2^{2/3} \approx 1.587,\ 2^{4/3} \approx 2.520,\ 2^{8/3} \approx 6.350$ and $2^{10/3} \approx 10.079$, respectively. These values are given in the last row of Tables 7.1–7.3. We see from Tables 7.1–7.3 that the theoretical estimate (6.5) of Theorem 6.1 is in good accordance with the actual convergence rate of the proposed method. The irregular behavior of the ratios ϱ_N in Tables 7.2–7.3 for $\varepsilon_N < 10^{-9}$ can be explained by round-off errors in the computations.

Example 2. Consider the following boundary value problem:

$$\left(D_*^{1.5} y\right)(t) = f(t, y(t)), \quad y(0) = -1,\ y'(1) = 1.9,\ 0 \le t \le 1, \tag{7.6}$$

where

$$f(t, y) := \frac{1}{2} y^2 - \frac{1}{2}\left(t^{1.9} - 1\right)^2 + \frac{\Gamma(2.9)}{\Gamma(1.4)} t^{0.4}.$$

The exact solution of this problem is $y^*(t) = t^{1.9} - 1$ (see Fig. 2(a)). Its derivative $z^*(t) := \left(D_*^{1.5} y^*\right)(t) = \frac{\Gamma(2.9)}{\Gamma(1.4)} t^{0.4}$.

Denote $z := \left(D_*^{1.5} y\right)$. Then a solution y^* of (7.6) can be presented in the form $y^* = Gz^* + Q$ where

$$(Gz)(t) := \left(J^{1.5} z\right)(t) - t\left(J^{0.5} z\right)(1), \qquad Q(t) := -1 + 1.9t,\ t \in [0,\ 1],$$

and z^* is a solution of the integral equation

$$z(t) = f\left(t, (Gz)(t) + Q(t)\right), \quad t \in [0,\ 1].$$

Using the change of variables $t = \tau^\rho,\ \tau \in [0,\ 1],\ \rho \ge 1$, and notations (7.2) we get that $y_\rho^* = \left(G_\rho z_\rho^*\right) + Q_\rho$ where

$$(G_\rho z_\rho)(\tau) := \left(J_\rho^{1.5} z_\rho\right)(\tau) - \tau^\rho \left(J_\rho^{0.5} z_\rho\right)(1), \quad Q_\rho(\tau) := -1 + 1.9\,\tau^\rho,\ \tau \in [0, 1],$$

and $z_\rho^*(\tau) = z^*(\tau^\rho)$ is a solution of the integral equation

$$z_\rho(\tau) = f\left(\tau^\rho, \left(G_\rho z_\rho\right)(\tau) + Q_\rho(\tau)\right), \quad \tau \in [0, 1]. \tag{7.7}$$

Approximations $z_{\rho,N}$ to the solution z_ρ^* of (7.7) we find in the form (4.12) with $b = 1$; the needed coefficients $\{c_{jk}\}$ we find from the nonlinear system (4.13) by Newton method. At that the same grid points $\{t_j\}$ and collocation points $\{t_{jk}\}$ as in Example 1 are used. After that the approximations $y_{\rho,N}$ to the solution y^* of problem (7.6) by the formula (4.15) are found.

In Tables 7.4 and 7.5 the errors ε_N calculated by (7.5) and their ratios $\varrho_N := \varepsilon_{N/2}/\varepsilon_N$, are presented. In Fig. 2(b) the graph of the error $y_N(t) - y^*(t)$ for (7.6) in the case $m = 3, r = 1, \rho = 4, N = 64$ is displayed.

Since for problem (7.6) $\alpha = 1.5,\ n_1 = 1$ and $\nu = 0.6$, it follows from (6.5) that, for $\rho \ge 1,\ r \ge 1$ and sufficiently large N, in case $m = 2$,

$$\varepsilon_N \le c \begin{Bmatrix} N^{-0.9\rho r} & \text{if} \quad 1 \le \rho r \le \dfrac{25}{9} \approx 2.78 \\ N^{-2.5} & \text{if} \quad \rho r \ge \dfrac{25}{9} \end{Bmatrix},$$

and in case $m = 3$,

$$\varepsilon_N \le c \begin{Bmatrix} N^{-0.9\rho r} & \text{if} \quad 1 \le \rho r \le \dfrac{35}{9} \approx 3.89 \\ N^{-3.5} & \text{if} \quad \rho r \ge \dfrac{35}{9} \end{Bmatrix}.$$

Due to these error estimates the ratios ϱ_N in the case $m = 2$ for $\rho r = 1,\ \rho r = 2$ and $\rho r \ge 2.78$ could be approximately $2^{0.9} \approx 1.87,\ 2^{1.8} \approx 3.48$ and $2^{2.5} \approx 5.66$, respectively. In the case $m = 3$ for $\rho r = 1,\ \rho r = 2,\ \rho = 3$ and $\rho r \ge 3.89$ the ratios ϱ_N could be approximately $2^{0.9} \approx 1.87,\ 2^{1.8} \approx 3.48,\ 2^{2.7} \approx 6.50$ and $2^{3.5} \approx 11.31$, respectively. These values are given in the last row of Tables 7.4 and 7.5. As we can see from these tables the actual convergence rate accords with the estimate (6.5). Even for $\rho = 1,\ 2$ and in the case $m = 3$ also for $\rho = 3$ the convergence is substantially faster than it is predicted by the estimate (6.5).

From the presented examples we see that the proposed algorithms for the numerical solution of nonlinear fractional initial and boundary value problems of the form (1.1)–(1.2) are effective and with high order accuracy. The obtained experimental convergence rates in Example 1 support the theoretical ones. In particular, it follows from Tables 7.1–7.3 that, in general, the attainable order of global convergence of the proposed algorithms on conditions of Theorem 6.1 cannot be improved. On the other hand, it follows from Tables 7.4 and 7.5 that for some special problems the actual convergence rate of approximate solutions may be better than the theoretical convergence rate given by Theorem 6.1.

Table 7.4
Numerical results for problem (7.6) in the case $m = 2$.

N	$r=1,\ \rho=1$		$r=1,\ \rho=2$		$r=1,\ \rho=3$		$r=1,\ \rho=4$	
	ε_N	ϱ_N	ε_N	ϱ_N	ε_N	ϱ_N	ε_N	ϱ_N
4	$1.44\cdot10^{-3}$		$3.84\cdot10^{-4}$		$6.47\cdot10^{-4}$		$3.05\cdot10^{-3}$	
8	$5.39\cdot10^{-4}$	2.68	$6.07\cdot10^{-5}$	6.33	$1.03\cdot10^{-4}$	6.28	$4.79\cdot10^{-4}$	6.38
16	$2.10\cdot10^{-4}$	2.56	$1.01\cdot10^{-5}$	5.99	$1.74\cdot10^{-5}$	5.91	$8.06\cdot10^{-5}$	5.94
32	$8.07\cdot10^{-5}$	2.60	$1.73\cdot10^{-6}$	5.84	$3.02\cdot10^{-6}$	5.77	$1.39\cdot10^{-5}$	5.78
64	$3.08\cdot10^{-5}$	2.62	$3.00\cdot10^{-7}$	5.77	$5.28\cdot10^{-7}$	5.72	$2.44\cdot10^{-6}$	5.72
128	$1.17\cdot10^{-5}$	2.63	$5.23\cdot10^{-8}$	5.74	$9.27\cdot10^{-8}$	5.69	$4.28\cdot10^{-7}$	5.70
256	$4.43\cdot10^{-6}$	2.64	$9.15\cdot10^{-9}$	5.72	$1.63\cdot10^{-8}$	5.69	$7.52\cdot10^{-8}$	5.69
		1.87		3.48		5.66		5.66

Table 7.5
Numerical results for problem (7.6) in the case $m = 3$.

N	$r=1,\ \rho=1$		$r=1,\ \rho=2$		$r=1,\ \rho=3$		$r=1,\ \rho=4$	
	ε_N	ϱ_N	ε_N	ϱ_N	ε_N	ϱ_N	ε_N	ϱ_N
4	$5.78\cdot10^{-4}$		$4.64\cdot10^{-6}$		$9.16\cdot10^{-6}$		$2.02\cdot10^{-5}$	
8	$2.14\cdot10^{-4}$	2.70	$2.92\cdot10^{-7}$	15.90	$7.29\cdot10^{-7}$	12.56	$1.63\cdot10^{-6}$	12.35
16	$8.06\cdot10^{-5}$	2.66	$3.74\cdot10^{-8}$	7.80	$6.16\cdot10^{-8}$	11.84	$1.40\cdot10^{-7}$	11.69
32	$3.05\cdot10^{-5}$	2.65	$6.40\cdot10^{-9}$	5.85	$5.32\cdot10^{-9}$	11.57	$1.22\cdot10^{-8}$	11.48
64	$1.15\cdot10^{-5}$	2.64	$1.04\cdot10^{-9}$	6.18	$4.57\cdot10^{-10}$	11.66	$1.06\cdot10^{-9}$	11.43
128	$4.36\cdot10^{-6}$	2.64	$1.60\cdot10^{-10}$	6.49	$3.45\cdot10^{-11}$	13.25	$8.61\cdot10^{-11}$	12.36
256	$1.65\cdot10^{-6}$	2.64	$2.09\cdot10^{-11}$	7.64	$3.41\cdot10^{-12}$	10.09	$2.34\cdot10^{-12}$	36.74
		1.87		3.48		6.50		11.31

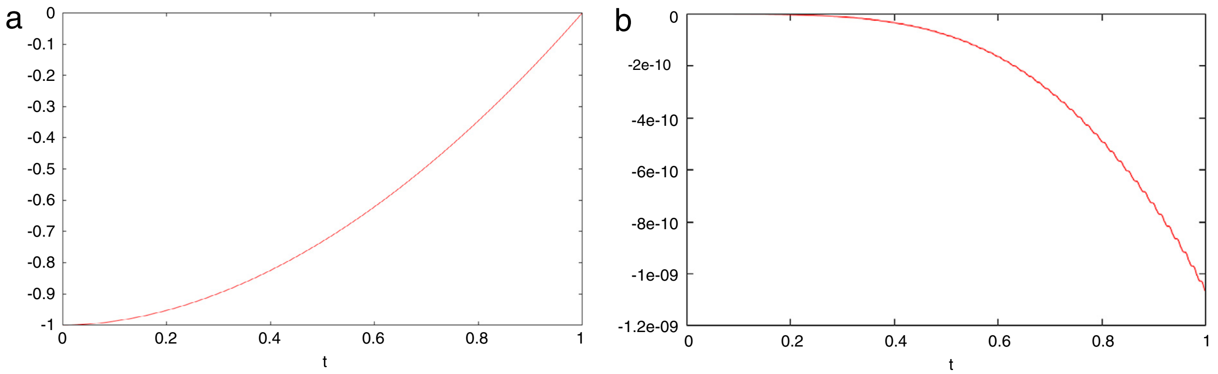

Fig. 2. (a) The exact solution $y^*(t) = t^{1.9} - 1$ for (7.6). (b) The error $y_N(t) - y^*(t)$ for (7.6) with $m = 3$, $r = 1$, $\rho = 4$, $N = 64$.

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