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Some remarks on the ball-covering property

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A B S T R A C T

A Banach space X has the so-called ball-covering property whenever its unit sphere can be covered by a countable collection of open balls that miss the origin. If $0 < \alpha < 1$, then X has the α-ball covering property if those balls miss αB_X. We show that X is separable if and only if X can be renormed such that the dual $(X, |||\cdot|||)^*$ enjoys the α-BCP for some (or all) $\alpha \in (0, 1)$. In contrast with this, we observe that the separability of X does not always imply the BCP of the dual X^*. The latter fact follows from a general example: non-separable $C(K)^*$ spaces fail the BCP in a major way — they even fail the "(-1)-BCP".

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1. Introduction

How many open balls are necessary to cover the unit sphere of $(\mathbb{R}^n, \|\cdot\|_2)$ if none of them contains the origin? Does the answer depend on n or the norm? What if the requirement is simply that none of them contains the closed unit ball? What are the infinite-dimensional analogues to the previous questions? Those and related questions —by the way, the three first are easy to answer— have been considered in several recent papers, a non-exhaustive list being [2,4–7,12,18].

Coverings of the closed unit ball B_X or the unit sphere S_X of a Banach space X are frequent in the literature. The context varies from finite-dimensional to infinite-dimensional Banach spaces and coverings range from finite to infinite —often countable—, from coverings by closed sets with non-empty interiors to ball-coverings. In the latter case, the balls may be, for example, required to be centered on S_X, to have bounded radii, to miss the origin, or to miss some prescribed multiple of B_X. A good account of all this can be found in [12] and [16].

Let us mention a fact concerning infinite-dimensional Banach spaces X, proved by several authors [16, Propositions 3 and 3ter]: *Any finite covering of S_X by balls contains the origin (in fact, it even covers B_X).*

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This may be one of the reasons for the introduction of the **ball-covering property** (**BCP**, in short) in [2] and subsequent papers: X is said to have the BCP if S_X can be covered by a countable collection of open balls, all of which miss the origin; such a collection is then called a countable **ball-covering** (off the origin). Later on, in [5], as an extension of this concept, countable *ball-coverings α-off the origin* ($\alpha > 0$) were introduced and studied in connection to the superreflexivity of the space.

Connections of the ball-covering property to the separability of the space X or the w^*-separability of the dual space X^* are relatively well understood. Apart from all separable spaces there are also some non-separable Banach spaces enjoying the BCP [5, Theorem 2.1]. On the other hand, it was proved in [2, Theorem 3.1 and Proposition 4.1] that *the BCP with balls of radii at most r $(0 < r < 1)$ implies the separability of the space*, and that *the BCP implies the w^*-separability of the dual space*. The converse of the latter is not true, as observed in [7]. However, in [6] and simultaneously in [12, Theorem 1.4] it was proved that it holds under renorming, namely that *the w^*-separability of X^* is equivalent to the fact that for every $\varepsilon > 0$ there is an equivalent $(1 + \varepsilon)$-equivalent norm $\|\cdot\|$ on X such that $B_{(X,\|\cdot\|)}$ can be covered with countably many $\|\cdot\|$-balls which do not contain* 0. Moreover, the result in [12] allows to choose the balls with a common radius. A similar renorming result relating the existence of countable ball-coverings α-off the origin of S_X and the existence of a separable norming subspace in X^* is also contained in [12].

The principal goal of the current note is to clarify the connection between the separability of the space X and the ball-covering properties of the dual. To have a more precise language, we define the *α-ball-covering property* $(-1 \le \alpha < 1)$ and related notions in Section 2. Section 4 contains positive results in the direction "separability implies the BCP of the dual", including the analogue (see Theorem 15) of the aforementioned renorming results. Section 5 provides a negative (isometric) result: $C[0, 1]$ is separable but $C[0, 1]^*$ fails the BCP (even the (-1)-BCP). This section happens to fully describe the ball-covering properties of $C(K)^*$-spaces and is viewed in the general setting of Banach lattices with an L-norm. The last section complements this view with a connection between the (-1)-BCP of the space X and the embeddability of $\ell_1(\Gamma)$ in X for some uncountable Γ.

For Section 4 we do need a couple of observations on the connection between the α-BCPs of the space and the w^*-separability or the existence of separable norming subspaces of the dual, under some differentiability conditions. These are provided in Section 3. That differentiability and ball-covering properties of the space are related has been considered already in [2], and then in [5,8,9,17], among others.

In this note we shall use the following notations/conventions: B_X (S_X) denotes the closed unit ball (respectively, the unit sphere) of a normed space X. Given a norm $\|\cdot\|$ on X, its dual norm will be denoted by $\|\cdot\|^*$ (or just by $\|\cdot\|$ if there is no risk of misunderstanding). The action of an element x^* of X^* on an element $x \in X$ will be denoted by $\langle x, x^* \rangle$ ("even" duals on the left, "odd" on the right). The open ball with center x and radius $r \ge 0$ will be denoted by $B(x;r) := \{y \in X : \|x - y\| < r\}$, and $\overline{B}(x;r) := \{y \in X : \|x - y\| \le r\}$ will denote the corresponding closed ball (warning: If $r = 0$, then $B(x;r)$ is the empty set, so $\overline{B}(x;0)$ *is not* the closure of $B(x;0)$). By a **smooth point** we mean a point of a Banach space where the norm is Gâteaux differentiable. This note deals with real vector spaces.

2. The α-ball-covering property and related notions

In the following, our context will be a seminormed space $(X, \|\cdot\|)$.

Note that in [5] and the subsequent works, the **ball-covering α-off the origin** for $\alpha > 0$ is defined as a collection of open balls covering S_X and such that none of them intersects the open ball $B(0, \alpha)$.

Here, we find it more convenient to speak about ball-coverings missing some closed ball $\alpha B_X = \overline{B}(0, \alpha)$, instead. That is, we say that a collection $\mathcal{B}$ of open balls is a **strictly α-off ball-covering (of S_X)** if $S_X \subset \bigcup \mathcal{B} \subset X \setminus \alpha B_X$.

Note that then this concept includes a ball-covering off the origin: It is just a strictly 0-off ball-covering. Also, as in the latter case, it is obvious that once you have infinite strictly α-off ball-coverings it does not

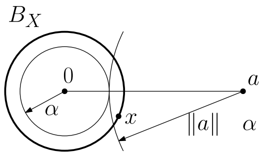

Fig. 1. Inequality (2) for $\alpha \geq 0$.

matter whether they consist of open or closed balls, because an open ball is a countable union of some smaller closed balls. However, open balls are somewhat simpler to deal with, because (if exact values of radii are not important) you can consider only one, the largest possible, radius per a ball center in your strictly α-off ball-covering.

Let us make explicit some related trivial formulae for the later use. Note that, for some $a \in X$, $r > 0$, and $\alpha \geq 0$,

$$\boxed{B(a;r) \cap \alpha B_X = \emptyset \text{ if, and only if, } \alpha \leq \|a\| - r.} \tag{1}$$

Given $\alpha \in \mathbb{R}$,

$$\boxed{\|a\| > \alpha \text{ and } x \in B(a, \|a\| - \alpha) \text{ if, and only if, } \|a\| - \|a - x\| > \alpha,} \tag{2}$$

(for $\|a\| > \alpha \geq 0$, see Fig. 1); and

$$\boxed{-1 \leq \|a\| - \|a - x\| \leq 1, \text{ for all } x \in S_X,\ a \in X.} \tag{3}$$

Equation (2) suggests the following.

Remark 1. Given $\alpha \in [0, 1)$ and $A \subset X$, it is possible to define a strictly α-off ball-covering with balls centered at the points from the set A if and only if

$$\boxed{\text{for every } x \in S_X \text{ there exists } a \in A \text{ with } \|a\| - \|a - x\| > \alpha.} \tag{4}$$

Indeed, if $\mathcal{B}$ is such a ball-covering, then every $x \in S_X$ is contained in some $B(a, r) \in \mathcal{B}$. So (1) gives $\|a - x\| < r \leq \|a\| - \alpha$.

Conversely, remove from A all elements a with $\|a\| \leq \alpha$ if necessary, and note that (4) means that $S_X \subset \bigcup_{a \in A} B(a; \|a\| - \alpha)$, so $\mathcal{B} := \{B(a; \|a\| - \alpha) :\ a \in A\}$ is a strictly α-off ball-covering. $\diamondsuit$

Recall that the BCP means that the space admits a countable ball-covering off the origin, that is, a strictly 0-off ball-covering. It is natural to define the **α-ball-covering property** as the existence of a countable strictly α-off ball-covering. By (3), this would make sense for all $\alpha \in [0, 1)$. But note that the equivalent condition (4) makes sense for all $\alpha \in [-1, 1)$. It turns out, as we will see later, that negative values of α also provide interesting notions, so we have the following.

Definition 2. Let $\alpha \in [-1, 1)$ and let $(X, \|\cdot\|)$ be a seminormed space. We say that the space $(X, \|\cdot\|)$ has the **α-ball covering property** (**α-BCP**, in short) whenever there exists a countable subset A of X such that,

$$\boxed{\text{for every } x \in S_X \text{ there exists } a \in A \text{ with } \|a\| - \|a - x\| > \alpha.} \tag{5}$$

In particular, the BCP is exactly the 0-BCP. Note that if $(X, \|\cdot\|)$ has the α-BCP for some $\alpha \in [-1, 1)$, then trivially it has the α'-BCP for any $\alpha' \in [-1, \alpha]$.

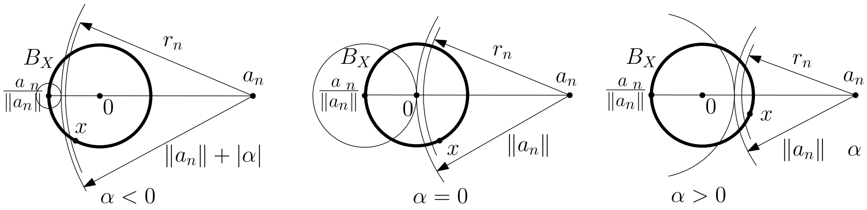

Fig. 2. Cases $\alpha < 0$, $\alpha = 0$, and $\alpha > 0$ in Definition 2.

Remark 3. Let us note in advance that while extending the definition of the α-BCP to negative values of α does not produce any isomorphically new concept (by Remark 13.2 every Banach space failing the BCP can be renormed to have the α-BCP for any $\alpha \in [-1, 0)$), it does describe some isometrically different classes of spaces. There are spaces: a) having the BCP, b) failing the BCP but having the α-BCP for any $\alpha \in [-1, 0)$ (see Remark 13.3), c) failing the (-1)-BCP (see Sections 5 and 6). $\diamondsuit$

Remark 4. An alternative formulation of the equivalence in (1), now valid in this larger setting, is the following: For every $\alpha \in [-1, 1)$, $r > 0$, and $a \neq 0$,

$$\boxed{B(a; r) \cap \overline{B}(-a/\|a\|; 1 + \alpha) = \emptyset \text{ if, and only if, } \alpha \leq \|a\| - r.} \tag{6}$$

Fig. 2 depicts the relative locations of the balls in (6) above for several α's.

To convey the geometrical meaning of Definition 2 we can thus extend the notion of a **strictly α-off ball-covering** to all $\alpha \in [-1, 1)$: It is a collection of open balls $\mathcal{B} = \{B(a_i, r_i)\}_i$ such that $S_X \subset \bigcup \mathcal{B}$ but $B(a_i, r_i) \cap \overline{B}(a_i/\|a_i\|; 1 + \alpha) = \emptyset$ for all i (see Fig. 2). Then the α-BCP again means exactly the existence of a countable strictly α-off ball-covering, for all $\alpha \in [-1, 1)$. $\diamondsuit$

However, for computations, it will be convenient to concentrate on sets of centers of the covering balls, and sometimes speak about coverings of the entire space, not only the sphere. Thus, we have the following.

Definition 5. Let $\alpha \in [-1, 1)$, let $(X, \|\cdot\|)$ be a seminormed space, and let $A, B \subset X$. We say that A is an **α-off net** for B if

$$B \setminus \{0\} \subset \{x \in X : \exists a \in A \ \|a\| - \|a - x\| > \alpha\|x\|\}.$$

To be concise, we will also denote the right-hand side above by $\mathcal{C}(\alpha, A)$.

Clearly, A is an α-off net for S_X if and only if there exists a strictly α-off ball-covering having ball centers in points of A. Definition 2 thus states that the space X has the α-BCP if and only if there exists a countable α-off net $A \subset X$ for S_X.

Definition 5 allows for some immediate but useful observations.

- The operation $(\alpha, A) \mapsto \mathcal{C}(\alpha, A)$ is non-decreasing with respect to both the set A and the number α.
- The continuity of the seminorm gives that $\mathcal{C}(\alpha, A) = \mathcal{C}(\alpha, \overline{A})$. In particular, the space X has the α-BCP if and only if there is a separable α-off net for S_X.
- If $A \subset X$ is an α-off net for S_X, then $\mathbb{Q}A$ is an α-off net for X. In particular, the space X has the α-BCP if and only if there is a countable (or a separable) set $A \subset X$ such that for every $x \in X \setminus \{0\}$ there is $a \in A$ with

$$\boxed{\|a\| - \alpha\|x\| > \|a + x\|.} \tag{7}$$

- As long as we do not care about the boundedness of an α-off net A, it is enough to consider the action of the one-sided directional derivatives of the norm at the points from A. Indeed, note that given $a, x \in X$, the quantity $\|\lambda a\| - \|\lambda a - x\|$ is increasing for $\lambda \in (0, \infty)$ and bounded from above by $\|x\|$. In fact, its limit is just the lower directional derivative of the norm at a:

$$d_a^-(x) := \lim_{t \to 0-} \frac{\|a + tx\| - \|a\|}{t} = \sup_{\lambda > 0} \|\lambda a\| - \|\lambda a - x\|. \tag{8}$$

  Clearly, $\mathcal{C}(\alpha, \mathbb{N}A)$ is the set of all $x \in X$ for which there exists $a \in A$ with $d_a^-(x) > \alpha\|x\|$.

Finally, for some arguments below it will be convenient to operate with the least upper bound of all possible constants α, for which a given space X has the α-BCP. So given a nonempty subset A of X, put

$$\boxed{bc(A, S_X) := \inf_{x \in S_X} \sup_{a \in A} \{\|a\| - \|a - x\|\}.} \tag{9}$$

Note that $bc(A, S_X)$ always exists and belongs to $[-1, 1]$ (see equation (3)). Put now

$$\boxed{bc(X) := \sup\{bc(A, S_X) : \ A \subset X, \ A \text{ countable}\}.} \tag{10}$$

Proposition 6. *Always, $bc(X) \geq -1$. If $bc(X) \in (-1, 1]$, then $bc(X) = \sup\{\alpha \in [-1, 1) : \ X$ has the α-BCP$\}$.*

Proof. The proof follows straight from the definitions. We write it down for the sake of completeness: The first statement follows from (3). For the second statement, fix $\alpha \in [-1, bc(X))$. Find a countable subset A of X such that $\alpha < bc(A, S_X)$. It follows that, for every $x \in S_X$, $\alpha < bc(A, S_X) \leq \sup_{a \in A}\{\|a\| - \|a - x\|\}$; thus, for every $x \in S_X$ there exists $a \in A$ such that $\alpha < \|a\| - \|a - x\|$. This shows, in view of (5), that X has the α-BCP.

The other direction follows from the fact that $bc(A, S_X) \geq \alpha$ whenever A is an α-off net for S_X. □

Example 7.

1. It is obvious that *every separable Banach space has the α-BCP for every $\alpha \in [-1, 1)$*. Put another way, $bc(X) = 1$ for any separable Banach space X (see Proposition 6).
   Indeed, if $\{x_n\}_{n=1}^{\infty}$ is a dense sequence in S_X, then $\mathcal{B} := \{B(x_n; 1 - \alpha) : \ n \in \mathbb{N}\}$ is an α-off covering of S_X. By using Mazur's theorem (see, e.g., [11, Theorem 8.2]), we may even assume that all centers of the balls in $\mathcal{B}$ are smooth points. This is related to Lemma 12 below.
2. There are non-separable Banach spaces X with $bc(X) = 1$. An example is the Banach space $(\ell_\infty, \|\cdot\|_\infty)$. Indeed, $bc(\ell_\infty) = bc(\{\pm 2e_n\}_{n=1}^{\infty}, S_{\ell_\infty})$ $(= 1)$ (this is essentially [2, Example 3.4]). Moreover, this is also true for every dual equivalent norm in ℓ_∞, since ℓ_1, being a separable dual space, has the Radon–Nikodým property, and we may apply Lemma 14 below.
3. The α-ball-covering property is not stable under renormings (it is, however, for separable spaces; see 1 above). Indeed, it was proven in [4, Theorem 2.1] that *for $\lambda \in (0, 1]$, the space $(\ell_\infty, \|\cdot\|_\lambda)$ has the BCP if, and only if, $\lambda > 1/2$, where $\|\cdot\|_\lambda$ is the equivalent norm*

$$\|\cdot\|_\lambda := \lambda\|\cdot\|_\infty + (1 - \lambda)p$$

   and p is the seminorm on ℓ_∞ given by $p : (x_n) \mapsto \limsup_n |x_n|$ for $(x_n) \in \ell_\infty$.
   From the proof, it can be seen that $bc((\ell_\infty, p)) = -1$. Moreover, this proof also provides that given $\lambda \in (0, 1)$, the equivalent norm $\|\cdot\|_\lambda$ defined above satisfies the inequality

$$bc((\ell_\infty, \|\cdot\|_\lambda)) \leq \lambda bc(\ell_\infty) + (1-\lambda)bc((\ell_\infty, p)) = 2\lambda - 1,$$

which becomes an equality whenever $\lambda > 1/2$.

According to Example 7.2 above, $\|\cdot\|_\lambda$ on ℓ_∞ cannot be a dual norm but in the trivial case $\lambda = 1$ (i.e., for $\|\cdot\|_\lambda = \|\cdot\|_\infty$).

4. In general we cannot conclude that X has the $bc(X)$-BCP (although, according to Proposition 6, if $bc(X) > -1$ then X has the α-BCP for any $-1 \leq \alpha < bc(X)$). The first thing to be noted in this direction is that $bc(X)$ exists in $[-1, 1]$ for any Banach space, although there are Banach spaces failing the α-BCP for any $\alpha \in [-1, 1)$ (and then $bc(X) = -1$ by Proposition 6). An example is $X := (\ell_1[0,1], \|\cdot\|_1)$ [7, Theorem 2.1]. See Section 5 below for a generalization of this result.

   The second thing is that we may have $bc(X) = 1$ (see above), although it makes no sense to ask for X to have the 1-BCP.

   For still another instance see Example 13.3 below.
5. None of the properties α-BCP, $\alpha \in [-1, 1)$, is hereditary. For example, $(\ell_\infty, \|\cdot\|_\infty)$ enjoys the 1-BCP (see 2 above), while $(\ell_1[0,1], \|\cdot\|_1)$, isometric to a subspace of $(\ell_\infty, \|\cdot\|_\infty)$ (see, e.g., [11, Exercise 3.103]), fails the (-1)-BCP, as mentioned in 4 above.
6. The Johnson–Lindenstrauss space has a renorming $\tilde{JL}$ that has the BCP but fails the α-BCP for any $\alpha > 0$ (see [12, p. 944]). So $bc(\tilde{JL}) = 0$.
7. Every rotund space has (at least) the (-1)-BCP. Indeed, let X be rotund, pick any point $a \in S_X$, and put $A = \{-a, a\} \subset X$. For any $x \in S_X$, if $\|a\| - \|a - x\| = -1$, which is the same as $\|a\| + \|-x\| = \|a - x\|$, then $a = -x$, so $\|-a\| - \|x - a\| = 1$. ♣

Let us finish this section with observations about stability, some of which will be needed in Section 6 below. Note that this extends [17, Theorem 1].

Proposition 8. *Let $\alpha \in [-1, 1)$ and let X and Y be non-zero Banach spaces. Then:*

(i) *$X \oplus_1 Y$ has the α-BCP if and only if X and Y both have the α-BCP.*
(ii) *$X \oplus_p Y$ has the α-BCP if X and Y both have the α-BCP, for any $p \in [1, \infty]$.*
(iii) *$X \oplus_\infty Y$ has the (-1)-BCP if at least one of X and Y has the (-1)-BCP.*
(iv) *$X \oplus_p Y$ always has the (-1)-BCP, for any $p \in (1, \infty)$.*

Proof. Where applicable, let A_X and A_Y denote the α-off nets for X and Y, respectively.

(i) It is clear that given an α-off net A for $X \oplus_1 Y$, the projection $p_X(A) = \{x : (x, y) \in A\} \subset X$ is an α-off net for X. On the other hand, the product $A_X \times A_Y$ is an α-off net for $X \oplus_1 Y$.

(ii) Consider first the case $p < \infty$. Given (x, y) with $x \neq 0$, find $a \in \mathbb{R} \cdot A_X$ with $\|a\| - \|x\| > \|a - x\| - (1-\alpha)\|x\| > 0$ and $b \in \mathbb{R} \cdot A_Y$ with $\|b\| - \|y\| \geq \|b - y\| - (1-\alpha)\|y\| > 0$. By increasing the norms of a and b if necessary, we can assume that $\|a\| = \lambda\|x\|$ and $\|b\| = \lambda\|y\|$ for some $\lambda > 1$. Then

$$\begin{aligned}\|(a, b)\|_p - \|(x, y)\|_p &= \|(\|a\| - \|x\|, \|b\| - \|y\|)\|_p \\ &> \|(\|a - x\| - (1-\alpha)\|x\|, \|b - y\| - (1-\alpha)\|y\|)\|_p \\ &\geq \|(a - x, b - y)\|_p - (1-\alpha)\|(x, y)\|_p,\end{aligned}$$

because the norm of ℓ_p^2 is strictly monotone. Thus, $(\mathbb{R} \cdot A_X) \times (\mathbb{R} \cdot A_Y)$ is an α-off net for $X \oplus_p Y$.

Given $(x, y) \in X \oplus_\infty Y$ with $\|x\| \geq \|y\|$ and $x \neq 0$, we can cover it by $\mathbb{N} \cdot A_X \times \{0\}$. Find $a \in A_X$ such that $\|a\| - \alpha\|x\| > \|a + x\|$. The same inequality also holds for any λa with $\lambda \geq 1$. We can choose $n \in \mathbb{N}$ such that $\|na + x\| \geq \|y\|$, so $\|(na, 0)\| - \alpha\|(x, y)\| > \|(na + x, y)\|$.

(iii) Assume that X has the (-1)-BCP. Due to the proof above, we only need to cover pairs (x, y) with $\|x\| < \|y\|$. Pick a non-zero $qa \in \mathbb{Q} \cdot A_X$ such that $\|x + qa\| \leq \|y\|$. Then $\|(qa, 0)\| + \|(x, y)\| > \|(x + qa, y)\|$.

(iv) It is enough to fix points $x_0 \in X \setminus \{0\}$ and $y_0 \in Y \setminus \{0\}$. Then, e.g., for any $(x, y) \in X \oplus_p Y$ with $y \neq 0$ by the strict convexity of the norm on ℓ_p^2 one has

$$\|(x_0, 0)\|_p + \|(x, y)\|_p > \|(\|x_0\| + \|x\|, \|y\|)\|_p \geq \|(x_0 + x, y)\|_p. \quad \square$$

3. The BCP and separability of the dual

The α-ball-covering property for $\alpha > 0$ is related —and some times equivalent— to the existence of a particular norming sequence of vectors in the space or in its dual. This has been made explicit, for example, in [12, Theorem 2.2]: *For a Banach space X and some $r > 0$, it is equivalent to say that X^* contains a countable r-norming set for X, and that for any $\varepsilon > 0$ there are a $(1 + \varepsilon)$-equivalent norm $|||\cdot|||$ on X and positive numbers R and δ such that $(X, |||\cdot|||)$ has the δ-BCP with balls of radius R.*

Here we shall present three lemmata in this direction: The first one is a general statement about a bound to a lower directional derivative, the second one guarantees the existence of an α-norming sequence in X^* as soon as X has the α-BCP —treating also the dual case—, and the third one uses differentiability to establish a kind of converse. The statements concern the actual norm of the given Banach space.

If X is a Banach space and $r \in (0, 1]$, we say that a set $A^* \subset S_{X^*}$ is **r-norming for some** $x \in X$ if $(1/r)\|x\| \leq \sup_{a^* \in A^*} \langle x, a^* \rangle$. In this case we also say that A^* **r-norms** x. Obviously, if $x \in X$ is r-normed by $N \subset X^*$, then it is possible to pick a countable subset of N still r-norming x. An **r-norming set** $A^* \subset S_{X^*}$ **(for X)** is a set that is r-norming for every $x \in X$. Despite a little inconsistency, we say that a linear subspace N of X^* is **r-norming for X** if its unit ball $B_N := B_{X^*} \cap N$ is an r-norming set for X.

Lemma 9. *If $a \in X$ is 1-normed by $A^* \subset S_{X^*}$, then*

$$\|a\| - \|a - x\| \leq d_a^-(x) \leq \sup_{a^* \in A^*} \langle x, a^* \rangle$$

for every $x \in X$.

Proof. It follows from $\langle x, a^* \rangle = \langle \lambda a, a^* \rangle + \langle x - \lambda a, a^* \rangle \geq \langle \lambda a, a^* \rangle - \|x - \lambda a\|$ for all $a^* \in A^*$, and by taking the supremum on $a^* \in A^*$ and $\lambda > 0$. $\square$

Lemma 10. *Let X be a Banach space with the α-BCP for some $\alpha \in [-1, 1)$. Let $N \subset X^*$ be a 1-norming subspace. Then there exists a countable subset A^* of S_N such that $\sup_{a^* \in A^*} \langle x, a^* \rangle > \alpha$ for every $x \in S_X$.*

Proof. Let A be a countable α-off net for S_X. Given $a \in A$ find a countable subset N_a^* of S_N that 1-norms a. The countable set $A^* := \bigcup_{a \in A} N_a^*$ 1-norms every $a \in A$.

Fix $x \in S_X$ and find $a \in A$ such that $\|a\| - \|a - x\| > \alpha$. Then, by Lemma 9,

$$\alpha < \|a\| - \|a - x\| \leq \sup_{a^* \in A^*} \langle x, a^* \rangle,$$

as we wanted to show. $\square$

Remark 11. Several statements (some of them well known) are easy consequences of Lemmata 9 and 10 above and the Separation Theorem:

1. If X has the BCP and N is a 1-norming subspace of X^*, then N is w^*-separable.

2. In particular, if X^* has the BCP, then X is separable.
3. If $bc(X) = \alpha > 0$, then X^* has a $\|\cdot\|$-separable α-*norming* subspace N.

   Proof. By Lemma 10, there exists a countable subset A^* of S_{X^*} with $\sup_{a^* \in A^*} \langle x, a^* \rangle \geq \alpha$ for every $x \in S_X$, and this shows the statement by taking $N := \overline{\text{span}}^{\|\cdot\|^*}(A^*)$. □

4. In particular, if $bc(X) = 1$, then (B_{X^*}, w^*) is separable, due to the fact that in such a case, and following the notation in the previous item, N is 1-norming, and so B_N is w^*-dense in B_{X^*}. ♢

The Šmulyan Lemma shows that $x \in S_X$ is a smooth point precisely when there exists a single element $x^* \in S_{X^*}$ such that $\langle x, x^* \rangle = 1$. Such a point x^* is said to be a w^***-exposed point of** B_{X^*} (exposed by x). By requesting that the centers of an α-off covering of S_X should be smooth points —there are plenty of those in separable Banach spaces, for example—, the following result enhances Lemma 10 turning it into an equivalence. This is the key to the results in the next section. It is somehow inspired by [18, Theorem 15], although our proof here dramatically reduces the arguments used there. Our result covers the aforementioned theorem in [18].

Lemma 12. *Let X be a Banach space and let $\alpha \in [-1, 1)$. The two following statements are equivalent:*

(i) *There exists a sequence $\{x_n\}_{n=1}^\infty \subset X$ of smooth points such that $\sup_n \|x_n - x\| - \|x\| > \alpha$ for all $x \in S_X$ (thus, X has the α-BCP with a countable α-off net for S_X consisting of smooth points).*

(ii) *There exists a sequence $\{x_n^*\}_{n=1}^\infty$ of w^*-exposed points of B_{X^*} such that $\sup_{n\in\mathbb{N}} \langle x, x_n^* \rangle > \alpha$ for all $x \in S_X$.*

Proof. It follows from (8) and Lemma 9 above. For proving (i)⇒(ii), just consider the sequence of Gâteaux derivatives of the norm at all points x_n, $n \in \mathbb{N}$. For proving (ii)⇒(i) put $\{x_n\}_{n=1}^\infty := \{m z_n : \ n, m \in \mathbb{N}\}$, where $z_n \in S_X$ satisfies $\langle z_n, x_n^* \rangle = 1$ for all $n \in \mathbb{N}$. Then $d_{z_n}^- = x_n^*$. □

Remark 13.

1. Without the assumption that the sequence $\{x_n^*\}_{n=1}^\infty$ in (ii), Lemma 12, consists of w^*-exposed points we cannot conclude that X has the α-BCP for some α —even the (-1)-BCP of the space X is not guaranteed. Indeed, as discussed in Example 7.4 above, $X := \ell_1[0, 1]$ fails the (-1)-BCP, although it can be seen easily that B_{X^*} is w^*-separable. Note that, then, there exists a sequence $\{x_n^*\}_{n=1}^\infty$ in S_{X^*} such that

$$\inf_{x\in S_X} \sup_{n\in\mathbb{N}} \langle x, x_n^* \rangle = 1.$$

   This example illustrates the relevance of the renorming result by [12] and [6], that we quoted in the introduction.
   Another approach to this is to observe that, would (ii)⇒(i) in Lemma 12 hold without the requirement that each x_n^* is w^*-exposed, then the α-BCP of X for some $\alpha \in (0, 1)$ will imply that X had, in any equivalent norm, the α'-BCP for some other $\alpha' \in (0, 1)$. This is false by the examples of equivalent norms in ℓ_∞ without the BCP discussed in Example 7.3 (see [4]).
2. The border-case $\sup_{n\in\mathbb{N}} \langle x, x_n^* \rangle \geq 0$ in (ii) of Lemma 12 (which is always satisfied once there are some w^*-exposed points) provides the following: *If $bc(X) < 0$, then the norm of X is nowhere smooth* (observe that every Banach space $(X, \|\cdot\|)$ can be endowed with an equivalent norm $|||\cdot|||$ having at least a point of Gâteaux smoothness. To see this, take a norm-attaining functional x_0^* in X^* with $\|x_0^*\| = 2$. The set $B := \text{conv}\,\{B_{(X^*,\|\cdot\|^*)}, x_0^*, -x_0^*\}$ is the closed unit ball of an equivalent dual norm $|||\cdot|||^*$ in X^*. Clearly,

x_0^* is a w^*-exposed point of $B_{(X^*,|||\cdot|||^*)}$, so by the Šmulyan Lemma the norm $|||\cdot|||$ has a point of Gâteaux smoothness).

3. There exists a Banach space X failing the BCP but such that $bc(X) = 0$. Indeed, let X be a non-separable reflexive space. It fails the BCP, because the BCP would imply the separability of the dual by Remark 11.2, and it does so in every equivalent norm. However, by the above, every Banach space X can be equivalently renormed as $(X, |||\cdot|||)$ such that $bc((X, |||\cdot|||)) \geq 0$. $\diamondsuit$

4. Separability and the BCP of the dual

In [18, Theorem 15] it is proved that Remark 11.2 can sometimes be reversed, namely: *if X is a separable Banach space with the Radon–Nikodým property (RNP, in short), then X^* has the BCP*. By Lemma 14 below one actually gets more: $bc(X^*) = 1$. This naturally suggests to ask whether just the separability of the space X is enough for ensuring that X^* has the α-BCP for some $\alpha \in (0, 1)$ or, less demanding, the BCP. The answer is no. An example is $X := C[0, 1]$ (see Remark 21 below). We present in Corollary 20 a general statement pertaining spaces $C(K)$.

However, the answer is positive under renorming. This is the content of Theorem 15, a result in the spirit of the following due to [12] and [6]: *Let X be a Banach space with a w^*-separable dual. Then, for every $\varepsilon > 0$ there exists a $(1 + \varepsilon)$-equivalent norm $\|\cdot\|_\varepsilon$ on X such that $(X, \|\cdot\|_\varepsilon)$ has the BCP*. In [12] the covering consists of balls with a common radius R that depends only on ε (and is independent of the Banach space X).

Recall that a **strongly exposed point** of B_{X^*} is an element $x^* \in S_{X^*}$ such that, for some $x \in S_X$, $\operatorname{diam} S(B_{X^*}, x, \delta) \to 0$ as $\delta \to 0$, where $S(B_{X^*}, x, \delta) := \{y^* \in B_{X^*} : \langle x, y^*\rangle \geq 1 - \delta\}$ is the section of B_{X^*} defined by $x \in S_X$ and $\delta > 0$.

Lemma 14. *Let X be a separable Banach space such that its closed unit ball B_X is the closed convex hull of its strongly exposed points. Then $bc(X^*) = 1$.*

Proof. A strongly exposed point of B_X is obviously a (strongly) w^*-exposed point of $B_{X^{**}}$. The collection of those points 1-norms the dual X^*. By separability we can find a countable subset A of them that still does, that is

$$\inf_{y^* \in S_{X^*}} \sup_{a \in A} \langle a, y^*\rangle = 1.$$

It is enough now to apply Lemma 12 to get the conclusion. $\square$

Lemma 14 is applicable to separable locally uniformly rotund (LUR, in short) or separable RNP spaces — in particular, to separable dual spaces (see, e.g., [11, Exercise 8.27 and Theorem 11.3]). It is the key to the following characterization.

Theorem 15. *Let $(X, |||\cdot|||)$ be a Banach space. Then, the following statements are equivalent:*

(i) *X is separable.*

(ii) *For every $\varepsilon > 0$, there exists an $(1 + \varepsilon)$-equivalent norm $\|\cdot\|$ on X such that $bc((X^*, \|\cdot\|^*)) = 1$.*

(iii) *There exists an equivalent norm $\|\cdot\|$ on X such that $(X^*, \|\cdot\|^*)$ has the BCP.*

Proof. (i)⇒(ii): If X is separable, a basic theorem of Kadets ensures the existence of an equivalent LUR norm $\|\cdot\|$ on X (see, e.g., [11, Theorem 8.1]). Using Asplund averaging it can be chosen to be arbitrarily close to the original norm (see, e.g., [10, p. 52]). Apply Lemma 14.

(ii)⇒(iii) is obvious and (iii)⇒(i) is Remark 11.2. $\square$

Remark 16. It is proved in [2, Theorem 4.5] and [3, Theorem 0.4] that every Banach space $(X, \|\cdot\|)$ such that $\|\cdot\|$ is LUR and (B_{X^*}, w^*) is separable has the BCP. In fact, a careful inspection of the proof there shows that under those assumptions much more is true, namely that X is separable (hence both $bc(X) = 1$ and $bc(X^*) = 1$ by Example 7.1 and Lemma 14). Indeed, if (B_{X^*}, w^*) is separable, then so is (S_{X^*}, w^*), because the dual norm is w^*-lower semicontinuous. Thanks to the Bishop–Phelps theorem we may select a w^*-dense sequence $(x_n^*) \subset S_{X^*}$ of norm-attaining functionals. Find $x_n \in S_X$ such that $\langle x_n, x_n^* \rangle = 1$ for all $n \in \mathbb{N}$. Take $y \in S_X$. Since $\{x_n^*\}_{n=1}^{\infty}$ is 1-norming, we can find a sequence $\{x_{n_k}^*\}$ such that $\langle y, x_{n_k}^* \rangle \to 1$. Observe that

$$1 \geq \left\| \frac{y + x_{n_k}}{2} \right\| \geq \left\langle \frac{y + x_{n_k}}{2}, x_{n_k}^* \right\rangle = \frac{\langle y, x_{n_k}^* \rangle + 1}{2} \to 1.$$

Since the norm is LUR at y, it follows that $x_{n_k} \xrightarrow{\|\cdot\|} y$. $\diamondsuit$

Remark 17. If we replace LUR with LUR-renormability in the remark above, then the conclusions are false. We may provide two examples of LUR-renormable nonseparable spaces with a w^*-separable dual unit ball.

The first is the Johnson–Lindenstrauss space JL_0. This space is isomorphic to $C(K)$ for a particular compact space K. This compact space is separable and scattered with height 3. By [10, Theorem 4.8], JL_0 is LUR-renormable. The dual unit ball of JL_0 is w^*-separable (see, e.g., [11, Theorem 14.54]). The space JL_0 is clearly nonseparable. In general, if (B_{X^*}, w^*) is separable, then X is separable if, and only if, (B_{X^*}, w^*) is monolithic (i.e., its density character is the same as its weight). Every compact monolithic and separable space is metrizable.

The second is $\ell_1[0, 1]$. As it was mentioned in Remark 13.1, its dual unit ball is w^*-separable. The space is nonseparable and LUR renormable, being the dual space to an Asplund space (see, e.g., [10, Chapter VII]). As it was already mentioned, this space fails the BCP. $\diamondsuit$

5. The BCP in $C(K)^*$

It is inspiring to consider the case of $X := \ell_1(\Gamma)$ first, where Γ is an uncountable set. For $\Gamma = [0, 1]$, it was presented in [7]. The argument there goes as follows: Assume that $\{B(x_n; r_n)\}_{n=1}^{\infty}$ is a countable ball-covering off origin. Since each x_n has a countable support, there exists $\gamma \in \Gamma$ such that $x_n(\gamma) = 0$ for all $n \in \mathbb{N}$. There exists $n \in \mathbb{N}$ such that $e_\gamma \in B(x_n; r_n)$. Then

$$r_n > \|e_\gamma - x_n\|_1 = \|e_\gamma\|_1 + \|x_n\|_1 \geq 1 + r_n,$$

a contradiction. So $(\ell_1(\Gamma), \|\cdot\|_1)$ fails the BCP.

The above argument essentially uses the fact that the *L-space* $\ell_1(\Gamma)$ is not *countably generated* when Γ is uncountable.

Recall that a Banach lattice is an **L-space** if the norm is additive on positive elements (see, e.g., [15, Definition 1.4.6]. Then the norm is also additive on disjoint elements. Let $A^\perp = \{y\colon\ a \perp y,\ \forall a \in A\}$ denote the disjoint complement of a set $A \subset X$. Recall that $A^{\perp\perp}$ is the smallest band containing A (see, e.g., [15, Proposition 1.2.7]). Note that a Banach lattice X is **countably generated** (i.e., X coincides with $A^{\perp\perp}$ for some countable set $A \subset X$) if and only if it has a **weak order unit** (X coincides with $\{u\}^{\perp\perp}$ for some $u \in X$) (see, e.g., [15, Remark after Proposition 2.3.1]).

This condition actually characterizes L-spaces with the (-1)-BCP. This is easy to see due to the following special case of (7).

Remark 18. A Banach space X fails the (-1)-BCP if, and only if, for every countable (or separable) set $A \subset X$ there is $x \in X \setminus \{0\}$ such that

$$\boxed{\|a\| + \|x\| = \|a + x\| = \|a - x\|, \text{ for all } a \in A.} \quad \diamondsuit \tag{11}$$

It is well known and easy to verify that in L-spaces the equalities in (11) mean exactly that $a \perp x$. Indeed, since in a Riesz space $x \perp y$ if and only if $|x + y| = |x - y|$, it follows from the fact that L-spaces have strictly monotone norm (see, e.g., [1, Theorem 8.12.3 and Definition 9.45]). Thus, we have observed the equivalence (i) $\Leftrightarrow$ (ii) of the following lemma.

Lemma 19. *Let X be an L-space. Consider the following conditions.*

(i) *X has the (-1)-BCP.*

(ii) *X has a weak order unit.*

(iii) *X is weakly compactly generated.*

Then (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii).

Proof. For (ii) $\Rightarrow$ (iii), see, e.g., the proof of [13, Theorem 17.18]. Let us provide a short proof for completeness. If X has a weak order unit u, then $\text{span}[-u, u]$ is order-dense in X (see, e.g., [15, Propositions 1.2.5-1.2.6]). Since every L-space has order-continuous norm, $\text{span}[-u, u]$ is actually norm-dense. But $[-u, u]$ is weakly compact whenever the norm is order-continuous (see, e.g., [15, Theorem 2.4.2]). $\square$

In fact, all the conditions in the above lemma are equivalent, see Corollary 25 below.

Corollary 20. *Let $X = C(K)^*$ for a compact Hausdorff space K. Then either X is separable and $bc(X) = 1$, or X is non-separable and fails the (-1)-BCP.*

Proof. The case when X is not weakly compactly generated follows from the above lemma. If $X = C(K)^*$ is weakly compactly generated, then $C(K)$ is Asplund (by [14]), K is scattered, and $C(K)^*$ is isometric to $\ell_1(\Gamma)$ for some Γ (see, e.g., [11, Theorems 14.24-14.25]). But $\ell_1(\Gamma)$ is non-separable if, and only if, Γ is uncountable, if, and only if, $\ell_1(\Gamma)$ does not have a weak order unit (by the beginning of the current section), if, and only if, $\ell_1(\Gamma)$ fails the (-1)-BCP (yet again by the above lemma). $\square$

Remark 21. In particular, separability does not imply the BCP of the dual: $C[0, 1]$ is separable but $C[0, 1]^*$ fails the (-1)-BCP. $\diamondsuit$

6. The BCP and copies of $\ell_1(\Gamma)$

Example 7.7 seems innocent, however it contains the seed that allows the arguments in Section 5. But further development can be done. Let us restate Remark 18 as follows.

Lemma 22. *Let X be a Banach space without the (-1)-BCP and let $Y \subset X$ be a separable subspace. Then there exists $x \in S_X$ such that $\|y + \lambda x\| = \|y\| + |\lambda|$ for every $y \in Y$ and every $\lambda \in \mathbb{R}$.*

Let Γ be any index set. Note that if a family of vectors $\{e_\gamma\}_{\gamma\in\Gamma}$ is "isometric" to the canonical basis of $\ell_1(\Gamma)$, that is, if $\|\sum_i \lambda_i e_{\gamma_i}\| = \sum_i |\lambda_i|\|e_{\gamma_i}\|$ for any $\gamma_1, \gamma_2, \ldots, \gamma_n \in \Gamma$ and $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}$, then $\overline{\text{span}}\{e_\gamma\}_{\gamma\in\Gamma}$ is isometric to $\ell_1(\Gamma)$.

Proposition 23. *Let X be a Banach space without the (-1)-BCP. Then X contains an isometric copy of $\ell_1(\Gamma)$ for some uncountable Γ.*

Proof. Let us use Zorn's lemma. Consider a poset $\mathcal{P}$ of all families $A \subset X$ isometric to the canonical basis of $\ell_1(A)$ (A is considered both as the family and its index set), ordered by inclusion. The poset $\mathcal{P}$ is not empty, because it contains all singletons $\{x\} \subset X$ with $x \neq 0$. For every chain in $\mathcal{P}$, it also clearly contains its union. So by Zorn's lemma $\mathcal{P}$ must contain a maximal element, let us call it Γ. This Γ is uncountable, because otherwise, due to the above lemma, we could find an element $e \in X \setminus \Gamma$ such that $\Gamma \cup \{e\}$ is still in $\mathcal{P}$, which would contradict the maximality of Γ. $\square$

Remark 24. Proposition 23 cannot be reversed in general (even for Banach lattices) as shown by the example $X = \ell_1(\Gamma) \oplus_p \mathbb{R}$ that has the (-1)-BCP, for any $p \in (1, \infty]$ and any Γ (see Proposition 8 above). $\diamondsuit$

However, Proposition 23 does provide a description of L-spaces failing the (-1)-BCP. Indeed, it is well known that an isometry between L-spaces preserves disjointness, so an isometric copy of $\ell_1(\Gamma)$ in an L-space X is actually a closed sublattice, which is necessarily complemented (see, e.g., [15, Theorem 2.7.11]). Since $\ell_1(\Gamma)$ (for uncountable Γ) is not weakly compactly generated, neither is X (see, e.g., [11, p. 576]), so X fails the (-1)-BCP by Lemma 19. Thus we can state the following improvement of Lemma 19, which is presumably well-known for the most part.

Corollary 25. *Let X be an L-space. The following are equivalent.*

(i) *X has the (-1)-BCP.*
(ii) *X has a weak order unit.*
(iii) *X is weakly compactly generated.*
(iv) *X does not contain an isometric copy of $\ell_1(\Gamma)$ for any uncountable Γ.*
(v) *X does not contain a complemented subspace isomorphic to $\ell_1(\Gamma)$ for any uncountable Γ.*

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