Tartu Ülikooli üliõpilaste matemaatikaolümpiaad

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1) Olgu $f_1, \ldots, f_n$ positiivsed pidevad funktsioonid lõigul $[0,1]$, kusjuures $\int_0^1 f_k(x)dx = a_k$, iga $k \in \{1, \ldots, n\}$ korral. Tõesta, et leidub $x \in [0,1]$ nii, et
$$f_1(x)f_2(x)\ldots f_n(x) \leq a_1 a_2 \ldots a_n.$$

2) Kas igal reaalmuutuja polünoomil p leidub esitus $p = q_1 - q_2$, kus q_1 ja q_2 on rangelt kasvavad reaalmuutuja polünoomid?

3) Olgu 2010×2010 maatriks A defineeritud võrdusega $A_{ij} = \operatorname{sgn}(j-i)$, s.t.
$$A = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 & 1 \\ -1 & 0 & 1 & \cdots & 1 & 1 \\ -1 & -1 & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & -1 & 0 \end{pmatrix}.$$
Olgu $\Lambda(A)$ maatriksi A kompleksete omaväärtuste hulk. Leia $\max\{|\lambda| : \lambda \in \Lambda(A)\}$.

4) Kas rida
$$\sum_{m\geq 1,\ n\geq 1,\ m\neq n} \frac{1}{mn\,|m-n|}$$
koondub?

5) Leia kõik ühikuga kommutatiivsed ringid $(A, +, \cdot)$, mille korral leidub täisarv $n > 0$ nii, et
$$x^{2^n+1} = 1 \quad \forall x \in A \setminus \{0\}.$$

6) Olgu $x : [0, \infty) \to \mathbb{R}$ diferentseeruv funktsioon, mis rahuldab võrdust
$$x'(t) = -2x(t)\sin^2 t + (2 - |\cos t| + \cos t) \int_{t-1}^{t} x(s) \sin^2 s \; ds$$
iga $t \in [1, \infty)$ korral. Tõesta, et x on tõkestatud ning $\lim_{t\to\infty} x(t) = 0$.

Solutions

1) Set $g_k(x) = \frac{f_k(x)}{a_k}$, then $\int_0^1 g_k(x)dx = 1$. The Cauchy inequality now gives

$$\int_0^1 \sqrt[n]{g_1 g_2 \dots g_n} dx \leq \int_0^1 \frac{g_1 + \dots + g_n}{n} dx = 1.$$

Hence, there is an $x \in [0,1]$ such that $g_1(x)g_2(x)\dots g_n(x) \leq 1$, i.e. $f_1(x)f_2(x)\dots f_n(x) \leq a_1 a_2 \dots a_n$.

2) Yes. Let P be the set of all polynomials having such a representation. First we show that $x^n \in P$ for $n = 0, 1, 2, \dots$.

This follows from $x^{2n+1} = 2x^{2n+1} - x^{2n+1}$ for $n \geq 0$, $x^0 = (x+1) - x$, and $x^{2n} = p_n(x) - q_n(x)$, where $p_n(x) = x^{4n-1} + x^{2n} + nx$ and $q_n(x) = x^{4n-1} + nx$ for $n \geq 1$ (observe that $p_n'(x) = (4n-1)x^{4n-2} + 2nx^{2n-1} + n = (3n-1)x^{4n-2} + n(x^{2n-1}+1)^2 > 0$ and $q_n'(x) = (4n-1)x^{4n-2} + n > 0$).

Now take $f, g \in P$, i.e. $f = p_f - q_f$ and $g = p_g - q_g$ for strictly monotonically increasing p_f, p_g, q_f, q_g. Then $f + g = (p_f + p_g) - (q_f + q_g) \in P$. Also for $\lambda > 0$ we have $\lambda f = (\lambda p_f) - (\lambda q_f) \in P$, for $\lambda < 0$ we have $\lambda f = (-\lambda q_f) - (-\lambda p_f) \in P$. Finally, for $\lambda = 0$ we have $0 = \lambda f = x - x \in P$.

So P is a linear subspace in a vector space of all polynomials, and P contains all the monomials x_n. Thus P is the set of all polynomials.

3) We have

$$\begin{aligned}
\det(\lambda I - A) &= \det \begin{pmatrix} \lambda & -1 & -1 & \cdots & -1 & -1 \\ 1 & \lambda & -1 & \cdots & -1 & -1 \\ 1 & 1 & \lambda & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & \lambda \end{pmatrix} \\
&= \det \begin{pmatrix} \lambda+1 & 0 & 0 & \cdots & 0 & -1 \\ 1-\lambda & \lambda+1 & 0 & \cdots & 0 & -1 \\ 0 & 1-\lambda & \lambda+1 & \cdots & 0 & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1-\lambda & \lambda \end{pmatrix}
\end{aligned}$$

by subtracting column 2 from column 1, then column 3 from column 2 etc. Furthermore by pulling out a factor of $\frac{1}{2}$ from the last column and then adding the first 2009 columns to the last we get:

$$\begin{aligned}\det(\lambda I - A) &= \frac{1}{2}\det\begin{pmatrix}\lambda+1 & 0 & 0 & \cdots & 0 & -2\\ 1-\lambda & \lambda+1 & 0 & \cdots & 0 & -2\\ 0 & 1-\lambda & \lambda+1 & \cdots & 0 & -2\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 1-\lambda & 2\lambda\end{pmatrix}\\ &= \frac{1}{2}\det\begin{pmatrix}\lambda+1 & 0 & 0 & \cdots & 0 & \lambda-1\\ 1-\lambda & \lambda+1 & 0 & \cdots & 0 & 0\\ 0 & 1-\lambda & \lambda+1 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 1-\lambda & \lambda+1\end{pmatrix}.\end{aligned}$$

Developing this determinant by the last column yields

$$\det(\lambda I - A) = \frac{1}{2}((1-\lambda)^{2010} + (\lambda+1)^{2010}).$$

Thus we have $\det(\lambda I - A) = 0$ if and only if $1 - \lambda = \zeta(\lambda + 1)$ where $\zeta^{2010} = -1$, i.e.

$$\lambda = \frac{1-\zeta}{1+\zeta}.$$

Now ζ must be a 4020th root of unity that is no 2010th root of unity, i.e. $\zeta = e^{2\pi ik/4020}$ for some odd integer $1 \le k \le 4019$. Hence,

$$\lambda = \frac{1-e^{\pi ik/2010}}{1+e^{\pi ik/2010}} = -i\tan(\pi k/4020).$$

Now clearly the modulus of λ is maximal when $\pi k/4020$ is closest to $\pi/2$, i.e. for $k = 2009$ and $k = 2011$. Thus

$$\max\{|\lambda| : \lambda \in \Lambda(A)\} = \tan(\frac{2009}{4020}\pi).$$

4) Let us prove that the series converges:

$$\sum_{m\ge1,\ n\ge1,\ m\ne n} \frac{1}{mn\,|m-n|} = 2\sum_{m\ge1,\ n>m} \frac{1}{nm(n-m)} = 2\sum_{m\ge1,\ k\ge1} \frac{1}{m(m+k)k} =$$

$$= 2\sum_{m=k} \frac{1}{mk(m+k)} + 2\sum_{m\ne k} \frac{1}{mk(m+k)} =$$

$$= 2\sum_{m=1}^{\infty} \frac{1}{2m^3} + 4\sum_{m<k} \frac{1}{mk(m+k)} \le \sum_{m=1}^{\infty} \frac{1}{m^3} + 4\sum_{m<k} \frac{1}{mk(k+1)} =$$

$$= \sum_{m=1}^{\infty} \frac{1}{m^3} + 4 \sum_{m=1}^{\infty} \frac{1}{m} \sum_{k=m+1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1} \right) =$$

$$= \sum_{m=1}^{\infty} \frac{1}{m^3} + 4 \sum_{m=1}^{\infty} \frac{1}{m(m+1)} < \infty.$$

5) Let $(A, +, \cdot)$ be such a ring. Then $-1 = 1$, hence $-x = x \ \forall x \in A$ and $2x = 0 \ \forall x \in A$.

Let us now observe that in the binomial formula

$$(x+1)^{2^n+1} = \sum_{i=0}^{2^n+1} \binom{2^n+1}{i} x^i$$

only the coefficients of x^{2^n+1}, x^{2^n}, x^1, and x^0 are odd. Indeed, the number $\binom{2^n}{k} = \frac{2^n}{k}\binom{2^n-1}{k-1}$ is even for all $k \in \{1, \ldots, 2^n - 1\}$. The (2^n+1)-th row (mod 2) in Pascal's triangle is therefore $1, 1, 0, 0, \ldots, 0, 1, 1$.

Thus, we have $x^{2^n+1} + x^{2^n} + x + 1 = (x+1)^{2^n+1} = 1$ for all $x \in A \setminus \{1\}$.

Since $x^{2^n+1} + 1 = 1 + 1 = 0$ for all $x \in A \setminus \{0\}$, it follows that $x^{2^n} + x = 1$ for all $x \in A \setminus \{0, 1\}$.

Next, let us remark that every nonzero element of A is invertible, that is $(A, +, \cdot)$ is actually a field. Therefore, the equation

$$x^{2^n} + x + 1 = 0, \ \ x \in A \setminus \{0, 1\}$$

has at most two solutions, that is A has at most four elements.

From this we deduce that A is either F_2 or F_4. Both verify $x^3 = 1$ for all $x \in A \setminus \{0\}$.

6) Let us consider the equation

$$x'(t) = -2x(t)a(t) + (2 - |\cos t| + \cos t) \int_{t-1}^{t} x(s)a(s) \ ds,$$

where $a(t) \geq 0$ is a continuous function. If $x(t)$ is a solution, then let us estimate the upper right derivative $D^+|x(t)|$ of $|x(t)|$:

$$\begin{aligned} D^+|x(t)| &\le -2a(t)|x(t)| + (2-|\cos t|+\cos t)\int_{t-1}^t a(s)|x(s)|\,ds \\ &= -2a(t)|x(t)| + 2\int_{t-1}^t a(s)|x(s)|\,ds \\ &\quad -(|\cos t|-\cos t)\int_{t-1}^t a(s)|x(s)|\,ds \\ &= -2\left(\int_{-1}^0 a(t)|x(t)|\,ds - \int_{-1}^0 a(t+s)|x(t+s)|\,ds\right) \\ &\quad -(|\cos t|-\cos t)\int_{t-1}^t a(s)|x(s)|\,ds \\ &= -2\frac{d}{dt}\int_{-1}^0\int_{t+s}^t a(u)|x(u)|\,duds \\ &\quad -(|\cos t|-\cos t)\int_{t-1}^t a(s)|x(s)|\,ds. \end{aligned}$$

Hence,

$$D^+(|x(t)|+2\int_{-1}^0\int_{t+s}^t a(u)|x(u)|\,duds) \le -(|\cos t|-\cos t)\int_{t-1}^t a(s)|x(s)|\,ds. \tag{1}$$

Since the right-hand side of (1) is nonpositive, we get that the function

$$|x(t)| + 2\int_{-1}^0\int_{t+s}^t a(u)|x(u)|\,duds \tag{2}$$

is decreasing, and so $|x(t)|$ is bounded.

For $k=0,1,\dots$, let $H_k = [(2k+1)\pi-1, (2k+1)\pi+1]$. We show that

$$\max_{t\in H_k}\int_{t-1}^t a(s)|x(s)|\,ds \to 0 \quad (k\to\infty). \tag{3}$$

If we suppose that on the contrary, (3) does not hold, then there is an $\alpha>0$ and a sequence $\{t_n\}$ such that $t_n \in \cup_{k=0}^\infty H_k$, $t_n\to\infty$, and

$$\int_{t_n-1}^t {}_n a(s)|x(s)|\,ds \ge \alpha.$$

Then at least one of the inequalities

$$\int_{t_n-1}^{t_n-1/2} a(s)|x(s)|\,ds \ge \alpha/2, \quad \int_{t_n-1/2}^{t_n} a(s)|x(s)|\,ds \ge \alpha/2$$

holds. Since $|\cos t| - \cos t > 0$ on the interval $[(2k+1)\pi - 3/2, (2k+1)\pi + 3/2]$, there exists a $\beta > 0$ such that at every point of an interval of length $1/2$ and containing t_n, we have

$$(|\cos t| - \cos t) \int_{t-1}^{t} a(s)\, |x(s)|\, ds > \beta.$$

However, this and (1) imply

$$|x(t)| + 2 \int_{-1}^{0} \int_{t+s}^{t} a(u)\, |x(u)|\, du ds \to -\infty \quad (t \to \infty),$$

which is impossible. Hence, (3) holds.

From (3) it follows that

$$\max_{t \in H_k} \int_{-1}^{0} \int_{t+s}^{t} a(u)\, |x(u)|\, du ds \to 0 \quad (k \to \infty). \tag{4}$$

Using (4) and the monotonicity and nonnegativity of the function in (2), we can see that to prove the existence of the limit of $x(t)$ at infinity it is enough to show that there is a sequence $\{t_n\}$ of points of the set $\cup_{k=0}^{\infty} H_k$ for which $t_n \to \infty$ and $x(t_n) \to 0$ as $n \to \infty$. If there was no such sequence, then there would be a $k_0 \in \mathbb{N}$ and a $\gamma > 0$ with the property that $|x(t)| > \gamma$ for every $t \in H_k$ and $k \geq k_0$. This, however, contradicts (3) for $a(t) = \sin^2 t$.