

Studying some connections between Algebra and Logic - an undergraduate research proposal

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1 Recalling propositional logic

A *propositional logic* comprises the following.

1. A countable set $\mathcal{P} = \{p_1, p_2, \dots\}$ of *propositional variables* (representing essentially *propositions*).
2. A set $\mathcal{L}$ of *connectives*, with an *arity* attached to each member of $\mathcal{L}$.
3. A set $\mathcal{F}_{\mathcal{L}}$ of *formulae*, constructed recursively from $\mathcal{P}$, using the connectives from $\mathcal{L}$.
4. A set of *axioms* $\mathcal{A}_{\mathcal{L}} \subset \mathcal{F}_{\mathcal{L}}$.
5. A set of *inference rules* (explained below), by which we mean an ordered pairs $\langle \Phi, \varphi \rangle$ where $\Phi \subset \mathcal{F}_{\mathcal{L}}$ and $\varphi \in \mathcal{F}_{\mathcal{L}}$.

The *classical propositional logic* (CPL) uses the following connectives.

Connective	Arity
$\neg$	1
$\wedge$	2
$\vee$	2
$\rightarrow$	2
$\leftrightarrow$	2
$\top$	0
$\perp$	0

(*Modal logic* extends the above list by adding unary connectives $\Box$ and $\Diamond$.)

If p_1, p_2, p_3 are propositional variables then

$$p_1, p_1 \rightarrow p_2, p_1 \vee (p_2 \wedge p_3), \neg p_2, \perp$$

are all formulae in CPL (let us allow parentheses and brackets while writing formulae to make them readable).

The following formulae are among the axioms of CPL.

1. $\varphi \vee \varphi \rightarrow \varphi$
2. $\varphi \rightarrow \varphi \vee \psi$
3. $\varphi \vee \psi \rightarrow \psi \vee \varphi$
4. $(\varphi \rightarrow \delta) \rightarrow [(\psi \rightarrow \delta) \rightarrow (\varphi \vee \psi \rightarrow \delta)]$
5. $\varphi \vee \neg\varphi$

The inference rule for CPL is

$$\langle \{\varphi, \varphi \rightarrow \psi\}, \psi \rangle \quad (\text{the } \textit{modus ponens}).$$

By a *substitution* we mean an assignment of formulae to variables.

For example, consider the CPL formula:

$$\varphi := ((p \vee q) \wedge \neg q) \rightarrow (p \wedge \neg q).$$

Then, the substitution $(p \rightarrow r)$ for p and $\neg q$ for q gives

$$\varphi' := (((p \rightarrow r) \vee \neg q)) \wedge \neg(\neg q)) \rightarrow ((p \rightarrow r) \wedge \neg(\neg q)).$$

We call the (new) formula, obtained after a substitution, a *substitution instance* of the original formula. In the above example φ' is a substitution instance of φ . In fact, a substitution can be seen as a function

$$\sigma : \mathcal{P} \longrightarrow \mathcal{F}_{\mathcal{L}},$$

i.e. from the set of variables to the set of formulae.

A formula ψ is said to be *directly derivable* from a set Ψ of formulae if there exists a substitution σ and an inference rule $\langle \Phi, \varphi \rangle$ such that

$$\sigma(\Phi) \subseteq \Psi \text{ and } \sigma(\varphi) = \psi.$$

For any $\Psi \subset \mathcal{F}_{\mathcal{L}}$ and any individual formula ψ , we write $\Psi \vdash_{\mathcal{L}} \psi$ if and only if ψ is contained in the smallest set of formulae containing Ψ together with all substitution instances of axioms and is closed under direct derivability by inference rules. Clearly, $\vdash_{\mathcal{L}}$ is a relation from the power set of $\mathcal{F}_{\mathcal{L}}$ to $\mathcal{F}_{\mathcal{L}}$. We call it the *consequence relation* of our logic. A subset Γ of $\mathcal{F}_{\mathcal{L}}$ is called a *theory* if $\Gamma \vdash_{\mathcal{L}} \psi$ implies that $\psi \in \Gamma$.

For any $\Phi \subset \mathcal{F}_{\mathcal{L}}$

$$\Gamma_{\Phi} = \{\varphi \in \mathcal{F}_{\mathcal{L}} \mid \Phi \vdash_{\mathcal{L}} \varphi\}.$$

is the set of all formulae that can be (syntactically) deduced from Φ . Clearly, Γ_{Φ} is the smallest theory containing Φ . We call Γ_{Φ} the *theory* of Φ . The formulas in Φ may be viewed as axioms of Γ_{Φ} . If Φ is understood, then we shall simply write Γ .

2 First-order logic

All of the above **syntactic machinery** can be (carefully) generalized to first-order logic (FOL) where, in addition to the *variables* we also have a set of *parameters* (essentially, constants) and where, in addition to the *connectives*, one has *quantifiers* (namely, $\forall$ and $\exists$) as well as *function* and *predicate* symbols (which may be viewed as relations or properties). An arity is attached to each function and each predicate symbol. The *formulae* of FOL are constructed inductively

- In the first step, *terms* are constructed inductively via operation symbols, using variables and parameters. (The variables may no longer be of propositional nature.)
- *Atomic formulae* are obtained by applying predicates to terms. Also $\top$ and $\perp$ are treated as atomic formulae.
- *Compound formulae* are constructed from the atomic ones using connectives.
- If φ is a formula and x is a variable, then $\forall x\varphi$ and $\exists x\varphi$ are called quantified formulae.

A typical formula in the (classical) first-order logic is

$$\forall x \forall y \exists z [R(f(x, z), y) \wedge R(x, g(y, z))], \quad (2.1)$$

where R is a binary predicate symbol and f, g are binary function symbols.

An occurrence of a variable in a first-order formula that falls within the *scope* of a quantifier is said to be *bound*. Otherwise, the occurrence is called *free*. A variable is said to be free in a formula φ if it has a free occurrence in φ . Otherwise, it is bound. All variables in (2.1) are bound, whereas in

$$[\forall x(P((f(x, y), g(y, z))) \rightarrow [\exists y(R(f(x, y), g(y, z))]$$

x, y, z are all free; indeed the third occurrence of x , the first and second occurrences of y and all occurrences of z are free. We write $\varphi(x_1, x_2, \dots, x_n)$ to tell that the free variables of a formula φ are among $x_1, x_2, \dots, x_n$. A formula with no free variable is called a *sentence*.

While carrying out a *substitution* in FOL we assign terms to free (occurrences of) variables. The inference rules for FOL include modus ponens and some further rules to take care of quantifiers (these are, essentially, $\langle \{\forall x\varphi(x)\}, \varphi(a) \rangle$ and $\langle \{\varphi(a)\}, \exists x\varphi(x) \rangle$). The consequence relation and theories for FOL are defined in a way similar to CPL.

3 Explicit definability

Let $P_1, P_2, \dots, P_K$ be predicates that occur in a theory Γ , and let P be an n -ary predicate in Γ . Then we say that P is *explicitly definable* from $P_1, P_2, \dots, P_K$

in Γ if there exists a formula $\varphi(x_1, x_2, \dots, x_n)$ whose predicates are among $P_1, P_2, \dots, P_K$ (i.e. φ does not use P) and

$$\Gamma \vdash_{\mathcal{L}} [P(x_1, x_2, \dots, x_n) \leftrightarrow \varphi(x_1, x_2, \dots, x_n)].$$

Example 3.1. Consider a theory Γ with predicate symbols $<$, $\leq$ and $=$, where the axioms of Γ are listed below.

1. $\forall x [\neg(x < x)]$
2. $\forall x \forall y \forall z [(x < y \wedge y < z) \rightarrow (x < z)]$
3. $\forall x \forall y [x \leq y \leftrightarrow (x < y \vee x = y)]$

Then taking $\varphi(x, y) = (x < y \vee x = y)$ one can immediately observe that $\varphi(x, y)$ explicitly defines $\leq$ from $<$ and $=$.

4 Implicit definability

In a theory Γ , a predicate P is said to be *implicitly defined* by (other) predicates $P_1, P_2, \dots, P_k$ if any two models $\mathcal{M}_1$ and $\mathcal{M}_2$ of Γ that agree on the *interpretation* of $P_1, P_2, \dots, P_k$ also agree on the interpretation of P .

It may be easier to understand the implicit definability with the help of an example where it fails.

Example 4.1. Consider a theory Γ , having only two binary predicates P and Fc . Assume that Γ has just one axiom:

$$\forall x \forall y [Fc(x, y) \rightarrow P(x, y)]. \quad (4.1)$$

Now, consider a model $\mathcal{M}_1$ of Γ whose universe is the registered residents of Tartu, and where $P(x, y)$ is interpreted as ' x is a parent of y ', and $Fc(x, y)$ is interpreted as ' y is the favorite child of x '. Then $\mathcal{M}_1$ clearly satisfies axiom (4.1), and is, therefore, a model of Γ . Let $\mathcal{M}_2 = \mathcal{M}_1$.

Consider a Tartu resident x who has two children y and z . Then $P(x, y)$ and $P(x, z)$ are interpreted the same way in both the models, that is each of them is true in $\mathcal{M}_1$ as well as $\mathcal{M}_2$. However, it does not violate the axiom (4.1) if $Fc(x, y)$ is true in $\mathcal{M}_1$ and false in $\mathcal{M}_2$ (equivalently, $Fc(x, z)$ is false in $\mathcal{M}_1$ and true in $\mathcal{M}_2$).

Thus Fc cannot be implicitly defined from P because there exist models $\mathcal{M}_1$ and $\mathcal{M}_2$ of Γ such that the interpretations of P agree in $\mathcal{M}_1$ and $\mathcal{M}_2$ but the interpretations of Fc does not.

Example 4.2. It can be shown that the predicate $\leq$ of Example 3.1 is also implicitly defined by $<$.

Remark 4.3. In fact, one can show that explicit definability implies the implicit one. But the converse is not always true.

5 Beth definability

A logic is said to have the *Beth definability property* if implicit definability in it implies explicit definability. Because the explicit and implicit definabilities are, respectively, syntactic and semantic in nature, Beth definability establishes a balance between the syntax and semantics. In other words, it says that a syntactic proof exists for what is semantically true.

6 Epimorphisms and dominions

A morphism $f : U \longrightarrow S$ in a category $\mathcal{C}$ is called an *epimorphism* shortly *epi* if

$$g \circ f = h \circ f \implies g = h$$

for all $\mathcal{C}$ -morphisms $g, h : S \longrightarrow T$ (one may say that f is right cancellative). Epis are category theoretic generalization of surjective homomorphisms. In particular, every surjective homomorphism in a variety of algebras (e.g. semigroups, monoids) is an epi, but the converse is not true (e.g. in the categories of all rings, all semigroups and all monoids epis are not necessarily surjective). A category $\mathcal{C}$ (given by a variety of algebras) is said to have the *Property ES* if epis are surjective in $\mathcal{C}$. The category of all groups is, for example, such.

Epis can also be defined via dominions. Given a subalgebra U of an algebra S , we say that $d \in S$ is dominated by U if for all homomorphisms $f, g : S \longrightarrow T$ that agree on U , one has $f(d) = g(d)$. The subalgebra of all elements of S dominated by U is called the *dominion* of U in S , denoted by $Dom_U S$. For a morphism $f : S \longrightarrow T$, one can easily verify that

$$f \text{ is an epi iff } Dom_{Im f} T = T.$$

7 Epimorphisms and Beth definability

An algebraizable logic has the Beth definability property if and only if the category of its corresponding algebras has the property *ES*.

8 Aim of the research project(s)

Our aim will be to understand the details of the above-mentioned connection between the surjectivity of epimorphisms and Beth definability. We may also try to see where the dominions fit in the picture. A Bachelor thesis in this area may provide grounds for studying the connections of surjective epimorphisms of ordered algebras with some kind of Beth definability. The latter may be a possible research topic for a Master thesis.