Surjectivity of epimorphisms and Beth definability - an undergraduate research proposal

25 September 2025

1 Introduction

A morphism $f : U \longrightarrow S$ in a category $\mathcal{C}$ is called an *epimorphism* shortly *epi* if

$$g \circ f = h \circ f \Longrightarrow g = h$$

for all $\mathcal{C}$-morphisms $g, h : S \longrightarrow T$ (one may say that f is right cancellative). Epis are category theoretic generalization of surjective homomorphisms. In particular, every surjective homomorphism in a variety of algebras (e.g. semigroups, monoids) is an epi, but the converse is not true (e.g. in the categories of semigroups and monoids epis are not necessarily surjective).

Epis can also be defined via dominions. Given a subalgebra U of an algebra S, we say that $d \in S$ is dominated by U if for all homomorphisms $f, g : S \longrightarrow T$, that agree on U, one has $f(d) = g(d)$. The 'subalgebra' of all elements of S dominated by U is called the *dominion* of U in S, denoted by $Dom_U S$. For a morphism $f : S \longrightarrow T$, one can easily verify that

$$f \text{ is an epi iff } Dom_{Imf} T = T.$$

Definitions 1.1. A subalgebra (e.g. submonoid) U of an algebra (e.g. monoid) is called *closed* if $Dom_U S = U$. We say that U is *absolutely closed* if it is closed in every overalgebra. We call a variety of algebras *absolutely closed* if each of its members is absolutely closed.

A category $\mathcal{C}$ (given by a variety of algebras) is said to have the *Property ES* if epis are surjective in $\mathcal{C}$. One can easily verify that $\mathcal{C}$ has the Property ES if and only if $\mathcal{C}$ is absolutely closed.

2 Research Proposal 2

The Property ES is closely related to a logical property known as *Beth definability*: a logic $\mathcal{L}$ is said to have the Beth definability property if implicit definability in

$\mathcal{L}$ implies explicit definability. Because the explicit and implicit definabilities have, respectively, syntactic and semantic flavor, Beth definability establishes a kind of balance between the syntax and semantics. An algebraizable logic $\mathcal{L}$ has the Beth definability property if and only if its corresponding Alg($\mathcal{L}$) has the property ES.

The aim of this research project is to understand connections between the surjectivity of epimorphisms and Beth definability as discussed in [1] and [2]. The Bachelor thesis may provide a ground for studying the connections of surjective epimorphisms of ordered algebras with some kind of Beth definability. The latter may be a possible research topic for a Master thesis.

References

[1] H. Andréka et al.: Algebraic Logic; Handbook of Philosophical Logic, Vol. II, 2nd Ed. pp. 133–247 (2001)

[2] E. Hoogland: Algebraic characterization of various Beth definability properties. Studia Logica 65(1), pp. 91–112 (2000)

[3] W. J. Blok and D. Pigozzi: Algebraizable Logics. AMS Memoirs No. 396 Volume 77 (1989)