Dominions and tensor products of monoids and pomonoids - an undergraduate research proposal

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1 Introduction

Let S be a monoid. By a *right S-act* A_S we mean a set A together with a right S-action $A \times S \longrightarrow A$ given by $(a, s) \longmapsto as$. A *left S-act* ${}_SA$ is defined analogously. In particular, every submonoid U of a monoid S gives the U-acts S_U and ${}_US$, where the action is just multiplication by the elements of U. If there is no chance of ambiguity, we shall denote A_S (${}_SA$) by A. Let A_S and ${}_SB$ be S-acts. We define

$$A \otimes_S B = (A \times B)/\tau,$$

where τ is the equivalence relation on $A \times B$ generated by the relation:

$$\{((as, b), (a, sb)) : a \in A, b \in B, s \in S\}.$$

We call $A \otimes_S B$ the *tensor product* of A_S and ${}_SB$.

A *posemigroup* S is a semigroup equipped with a partial order $\leq$ such that

$$s_1 \leq s_2,\ t_1 \leq t_2 \Longrightarrow s_1t_1 \leq s_2t_2$$

for all $s_1, s_2, t_1, t_2 \in S$. *Pomonoids* are defined similarly. Given a pomonoid S, a *right S-poset* is a poset A that is a right S-act such that, with the 'pointwise order',

$$(a_1, s_1) \leq (a_2, s_2) \implies a_1s_1 \leq a_2s_2,$$

for all $a_1, a_2 \in A$ and $s_1, s_2 \in S$. A right S-poset will also be denoted by A_S. *Left S-posets* may be defined similarly. The *Tensor product* of S-postes A_S and ${}_SB$ is

$$A \hat{\otimes}_S B = (A \times B)/\theta,$$

where θ is the smallest equivalence relation on $A \times B$ such that $(A \times B)/\theta$ is a poset with

$$(a_1, b_1) \leq (a_2, b_2) \implies (a_1, b_1)\theta \leq (a_2, b_2)\theta,$$

and where $(x, y)\theta$ denotes the equivalence class of (x, y). The elements of $A \otimes_S B$ and $A \hat{\otimes}_S B$ will respectively be denoted by $a \otimes b$ and $a \hat{\otimes} b$.

A *posemigroup homomorphism* $f : U \longrightarrow T$ is a homomorphism of the underlying semigroups such that the *monotonicity condition*, viz.

$$u_1 \leq u_2 \Longrightarrow f(u_1) \leq f(u_2)$$

for all $u_1, u_2 \in U$, is satisfied. A *Pomonoid homomorphism* is just a homomorphism of posemigroups. Clearly, not every homomorphism of the underlying semigroups is a homomorphism of the posemigroups.

Given a submonoid U of a monoid S, we say that $d \in S$ is *dominated* by U if for all homomorphisms $f, g : S \longrightarrow T$, that agree on U, one has $f(d) = g(d)$. The 'submonoid' of all elements of S dominated by U is called the *dominion* of U in S, denoted by $Dom_U S$. Dominions of pomonoids are defined in a similar fashion.

2 Research Proposal 1

It follows from what is explained in the introduction that one may begin by considering two kinds of dominions in the categories of posemigroups and pomonoids: by considering or ignoring the orders. They are characterized by the following criteria, known as zigzag theorems,

1. $d \in Dom_U S$ iff $d \otimes 1 = 1 \otimes d$, for monoids,
2. $d \in \hat{Dom}_U S$ iff $d \hat{\otimes} 1 = 1 \hat{\otimes} d$, for pomonoids,

where the tensor products are evaluated in the respective categories. It turns out [3] that $d \otimes 1 = 1 \otimes d$ if and only if $d \hat{\otimes} 1 = 1 \hat{\otimes} d$. The aim of this project is show that $x_1 \hat{\otimes} y_1 = x_2 \hat{\otimes} y_2$ does not imply $x_1 \otimes y_1 = x_2 \otimes y_2$ in general. (In fact $x_1 \otimes y_1 = x_2 \otimes y_2$ always implies $x_1 \hat{\otimes} y_1 = x_2 \hat{\otimes} y_2$.)

References

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[3] S. Nasir. Epimorphisms, dominions and amalgamation in pomonoids, Semigroup Forum 90, 800–809 (2015)