Proposal 2 - Studying the Amalgamation and Related Properties for Quantales

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1 Introduction to some background work

- A *semigroup* is a non-empty set S together with an associative binary operation.
- A *monoid* is a semigroup with identity.
- *Subsemigroups* and *submonoids*, as well as *oversemigroup* and *overmonoids*, are defined in the standard way.
- If X is a non-empty set then the set $\mathcal{T}(X)$ of all transformations of X is a monoid under the binary operation of the usual composition of maps. It is called the *full transformation monoid* over X. The identity transformation plays the role of identity in $\mathcal{T}(X)$.
- Given a semigroup (monoid) S and a non-empty set X, a homomorphism $\rho : S \to \mathcal{T}(X)$ is called a *representation* of S by $\mathcal{T}(X)$. For all $s \in S$, we denote $\rho(s)$ by ρ_s.
- A *(right) congruence* on semigroup (monoid) S is an equivalence relation on S that is '(right) compatible' with the binary operation.

Definition 1.1. Let S be a monoid and A be a non-empty set. Then A is called a *right S-act* if there exists a right S-action $(a, s) \longmapsto as$, $a \in A$, $s \in S$ such that,

1. for all $a \in A$, $s, t \in S$, $(as)t = a(st)$;
2. $a1 = a$, where 1 is the identity of S.

The notation A_S will mean that A is a right S-act. A *left S-act* ${}_SB$ is defined similarly. Any monoid S can naturally be made into the right S-act S_S as well as left S-act ${}_SS$. In general, every submonoid U of a monoid S gives rise to the U-acts S_U and ${}_US$.

- If S is a monoid then every representation $\rho : S \to \mathcal{T}(X)$ turns X into a right S-act, where for all $x \in X$ and $s \in S$,
$$xs = \rho_s(x).$$
- Conversely, every right S-act X_S gives rise to a representation $s \mapsto \rho_s$ of S by $\mathcal{T}(X)$, where for each $s \in S$ we define $\rho_s : X \to X$ by
$$\rho_s(x) = xs.$$
- One may dually define the same properties for left S-acts.
- Every monoid S has a representation $\rho : S \to S_S$, defined by $\rho(s) = \rho_s$, where $\rho_s(t) = ts$ for all $t \in S$. This is called the *right regular representation* of S.

Definition 1.2. Let S be a submonoid of a monoid T. Then S is said to have the *representation extension property* (REP) in T if for every representation $\rho : S \to \mathcal{T}(X)$ there exists a set $Y \supseteq X$ and a representation $\sigma : T \to \mathcal{T}(Y)$, given by $\sigma(t) = \sigma_t$, $t \in T$, such that
$$\sigma_s(x) = \rho_s(x)$$
for all $s \in S$ and $x \in X$. We say that S has representation extension property (REP) if it has (REP) in every overmonoid.

The following proposition defines (REP) in terms of the tensor product of S-acts. (Given two S-acts A_S and ${}_SB$, their tensor product $A \otimes_S B$ is the factor of $A \times B$ by certain equivalence, which we will not describe here.)

Proposition 1.3. *A submonoid U of a monoid S has (REP) in S if for every right U-act X_U the map*
$$X \to X \otimes_U S \text{ given by } x \mapsto x \otimes 1,$$
is one-to-one.

Definition 1.4. Let S be a submonoid of a monoid T. Then S is said to have the *right congruence extension property* (RCEP) in T if for every right congruence θ on S there exists a right congruence Θ on T such that
$$s_1 \Theta s_2 \text{ iff } s_1 \theta s_2,$$
for all $s_1, s_2 \in S$. We say that S has the right congruence extension property (RCEP) if it has (RCEP) in every overmonoid.

Definition 1.5. A right S-act A_S is called *flat* if for all left S-act embeddings
$$\phi : {}_SX \to {}_SY$$
the induced map
$$1 \otimes \phi : A \otimes_S X \to A \otimes_U Y$$
is one-to-one. We call S *right absolutely flat* (RAF) if all its right acts are flat. *Left absolute flatness* is defined similarly. We call S *absolutely flat* if it is both right and left absolutely flat.

Remark 1.6. We have the following implications

$$(RAF) \Rightarrow (REP) \Rightarrow (RCEP).$$

Definition 1.7. A semigroup (monoid) *amalgam* is a list $\mathcal{A} = (U; S_1, S_2)$ of semigroups (monoids) with $S_1 \cap S_2 = U$. We say that $\mathcal{A}$ is embeddable if there exists a semigroup (monoid) T with S_1 and S_2 being embedded in T such that

1. the embedding monomorphisms $\phi_i : S_i \to T, 1 \leq i \leq 2$, agree on U,
2. and only U.

If only Condition (1) is satisfied then $\mathcal{A}$ is *weakly embeddable*.

Theorem 1.8. *If U is absolutely flat then the amalgam $(U; S_1, S_2)$ is embeddable for all S_1, S_2, i.e. U is an amalgamation base in the class of all semigroups.*

Example 1.9. Inverse semigroups (which are not defined here) are absolutely flat, and hence amalgamation bases.

2 Research Project - Studying the Amalgamation and Related Properties for Quantales

A *sup-lattice* is a partially ordered set L in which all subsets have a supremum (or *join*). The join of $M \subseteq Q$ is denoted by $\vee M$. The partial order on L is defined by,

$$s_1 \leq s_2 \text{ iff } s_1 \vee s_2 = s_1, \; \forall \, s_1, s_2 \in L.$$

Of course, every sup-lattice is a complete lattice, in the sense that it also has infimums (or *meets*) for all subsets. However, the homomorphisms of sup-lattices differ from those of complete lattices, as they preserve only joins (but not necessarily meets).

Definition 2.1. By a *quantale* we mean a triple $(Q, \cdot, \vee)$, such that

1. $(Q, \cdot)$ is a semigroup,
2. $(Q, \vee)$ is a sup-lattice,
3. $x \cdot (\vee M) = \vee(x \cdot M)$ and $(\vee M) \cdot x = \vee(M \cdot x)$ for all $x \in Q$ and all $M \subseteq Q$.

A quantale is called *unital* if $(Q, \cdot)$ is a monoid.

Quantales can be represented by modules of sup-lattices. More precisely, given a quantale Q and a sup-lattice M, we say that *M is a right Q-module* if there exists a right action $(x, a) \mapsto xa$, where $x \in L$ and $a \in Q$, such that $\forall \, x \in L$,

1. $(xa)b = x(ab), \; \forall \, a, b \in Q$,
2. the action is distributive over arbitrary joins in both the coordinates,

3. $x1 = x$ if Q is a unital quantale with identity 1.

We write L_Q to mention that that L is a right Q-module. Left Q modules are defined analogously. As in the case of S-acts and S-posets, every L_Q may be viewed as representation

$$Q \to \text{End}(M),$$

and vice versa (indeed $\text{End}(M)$ is a unital quantale). The aim of this project is to answer the following question.

Question 3.2. How to define (REP), (RCEP), and (RAF) for qualtales and how are they related to each other and with the amalgamation properties?

Quantales have turned out to be important mathematical structures with applications in several branches of mathematics, especially Linear and Quantum Logic. They are also related with 'locales' (and 'frames') that may be viewed as 'pointless topologies'. Answering the above questions will provide useful insight into these structures.

References

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