Proposal 1 - Studying absolute flatness and amalgamation for inverse pomonoids

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1 Introduction to some background work

- A *semigroup* is a non-empty set S together with an associative binary operation.
- A *monoid* is a semigroup with identity.
- *Subsemigroups* and *submonoids*, as well as *oversemigroup* and *overmonoids*, are defined in the standard way.
- If X is a non-empty set then the set $\mathcal{T}(X)$ of all transformations of X is a monoid under the binary operation of the usual composition of maps. It is called the *full transformation monoid* over X. The identity transformation plays the role of identity in $\mathcal{T}(X)$.
- Given a semigroup (monoid) S and a non-empty set X, a homomorphism $\rho : S \to \mathcal{T}(X)$ is called a *representation* of S by $\mathcal{T}(X)$. For all $s \in S$, we denote $\rho(s)$ by ρ_s.
- A *(right) congruence* on semigroup (monoid) S is an equivalence relation on S that is '(right) compatible' with the binary operation.

Definition 1.1. Let S be a monoid and A be a non-empty set. Then A is called a *right S-act* if there exists a right S-action $(a, s) \longmapsto as$, $a \in A$, $s \in S$ such that,

1. for all $a \in A$, $s, t \in S$, $(as)t = a(st)$;
2. $a1 = a$, where 1 is the identity of S.

The notation A_S will mean that A is a right S-act. A *left S-act* ${}_SB$ is defined similarly. Any monoid S can naturally be made into the right S-act S_S as well as left S-act ${}_SS$. In general, every submonoid U of a monoid S gives rise to the U-acts S_U and ${}_US$.

- If S is a monoid then every representation $\rho : S \to \mathcal{T}(X)$ turns X into a right S-act, where for all $x \in X$ and $s \in S$,

$$xs = \rho_s(x).$$

- Conversely, every right S-act X_S gives rise to a representation $s \mapsto \rho_s$ of S by $\mathcal{T}(X)$, where for each $s \in S$ we define $\rho_s : X \to X$ by

$$\rho_s(x) = xs.$$

- One may dually define the same properties for left S-acts.
- Every monoid S has a representation $\rho : S \to S_S$, defined by $\rho(s) = \rho_s$, where $\rho_s(t) = ts$ for all $t \in S$. This is called the *right regular representation* of S.

Definition 1.2. Let S be a submonoid of a monoid T. Then S is said to have the *representation extension property* (REP) in T if for every representation $\rho : S \to \mathcal{T}(X)$ there exists a set $Y \supseteq X$ and a representation $\sigma : T \to \mathcal{T}(Y)$, given by $\sigma(t) = \sigma_t$, $t \in T$, such that

$$\sigma_s(x) = \rho_s(x)$$

for all $s \in S$ and $x \in X$. We say that S has representation extension property (REP) if it has (REP) in every overmonoid.

The following proposition defines (REP) in terms of the tensor product of S-acts. (Given two S-acts A_S and ${}_SB$, their tensor product $A \otimes_S B$ is the factor of $A \times B$ by certain equivalence, which we will not describe here.)

Proposition 1.3. *A submonoid U of a monoid S has (REP) in S if for every right U-act X_U the map*

$$X \to X \otimes_U S \text{ given by } x \mapsto x \otimes 1,$$

is one-to-one.

Definition 1.4. Let S be a submonoid of a monoid T. Then S is said to have the *right congruence extension property* (RCEP) in T if for every right congruence θ on S there exists a right congruence Θ on T such that

$$s_1 \Theta s_2 \text{ iff } s_1 \theta s_2,$$

for all $s_1, s_2 \in S$. We say that S has the right congruence extension property (RCEP) if it has (RCEP) in every overmonoid.

Definition 1.5. A right S-act A_S is called *flat* if for all left S-act embeddings

$$\phi : {}_SX \to {}_SY$$

the induced map

$$1 \otimes \phi : A \otimes_S X \to A \otimes_U Y$$

is one-to-one. We call S *right absolutely flat* (RAF) if all its right acts are flat. *Left absolute flatness* is defined similarly. We call S *absolutely flat* if it is both right and left absolutely flat.

Remark 1.6. We have the following implications

$$(RAF) \Rightarrow (REP) \Rightarrow (RCEP).$$

Definition 1.7. A semigroup (monoid) *amalgam* is a list $\mathcal{A} = (U; S_1, S_2)$ of semigroups (monoids) with $S_1 \cap S_2 = U$. We say that $\mathcal{A}$ is embeddable if there exists a semigroup (monoid) T with S_1 and S_2 being embedded in T such that

1. the embedding monomorphisms $\phi_i : S_i \to T, 1 \leq i \leq 2$, agree on U,
2. and only U.

If only Condition (1) is satisfied then $\mathcal{A}$ is *weakly embeddable*.

Theorem 1.8. *If U is absolutely flat then the amalgam $(U; S_1, S_2)$ is embeddable for all S_1, S_2, i.e. U is an amalgamation base in the class of all semigroups.*

Example 1.9. Inverse semigroups (which are not defined here) are absolutely flat, and hence amalgamation bases.

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A *posemigroup* $(S, \cdot, \leq)$, is a semigroup $(S, \cdot)$ endowed with a 'compatible' partial order $\leq$. A *pomonoid* is a posemigroup with identity. Given a pomonoid S, a *right S-poset* is a poset A admitting a right S-action ρ, as defined earlier, such that A_S is a right S-act and for all $a_1, a_2 \in A$ and $s_1, s_2 \in S$, if $(a_1, s_1) \leq (a_2, s_2)$ in $A \times S$ then,

$$(a_1, s_1)\rho \leq (a_2, s_2)\rho.$$

Pomonoids (posemigroups) are represented by S-posts, like monoids (semigroups) are represented by S-acts. One may define (REP), (RCEP) and (RAF) for pomonoids (posemigroups). The implications

$$(RAF) \Rightarrow (REP) \Rightarrow (RCEP).$$

also hold in the 'categories' of all posemigroups and pomonoids. However, unlike monoids, the inverse pomonoids are not always flat. Also, (RAF) may not imply the amalgamation property (as it does in the unordered case). The aim of this research project is to answer the following question.

Question 2.2. Are absolutely flat inverse pomonoids (posemigroups) amalgamation bases in the category of all pomonoids (posemigroups)?

Posemigroups (pomonoids) are important mathematical structures, and very little is known about their amalgamation properties. This project will provide important insight into the said properties.

References

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