Dominions and Epimorphisms for Pomonoids - Undergraduate Research Proposals

23 September 2024

1 Introduction and background

A homomorphism $f : U \to S$ of semigroups (monoids) is called an *epimorphism* shortly *epi* if

$$g \circ f = h \circ f \Rightarrow g = h$$

for all homomorphisms $g, h : S \to T$. Epis are category theoretic generalization of surjective homomorphisms. In particular, every surjective homomorphism (of semigroups, monoids) is an epi, but the converse is not true (in the categories of semigroups and monoids).
A *posemigroup* U is a semigroup equipped with a partial order $\leq$ such that

$$u_1 \leq u_2,\, v_1 \leq v_2 \Rightarrow u_1 v_1 \leq u_2 v_2$$

for all $u_1, u_2, v_1, v_2 \in U$. *Pomonoids* are defined similarly. A *posemigroup homomorphism* $f : U \to T$ is a homomorphism of the underlying semigroups such that

$$u_1 \leq u_2 \Rightarrow f(u_1) \leq f(u_2)$$

for all $u_1, u_2 \in U$. A *Pomonoid homomorphism* is just a homomorphism of semigroups.

Clearly, not every homomorphism of the underlying semigroups is a homomorphism of the posemigroups. However, it was proved in [3] that epis of pomonoids coincide with the epis of their underlying monoids.

Closely related to the concept of epis is the notion of *dominion*. Given a subsemigroup U of a semigroup S, we say that $d \in S$ is dominated by U if for all homomorphisms $f, g : S \to T$, that agree on U, one has $f(d) = g(d)$. The 'subsemigroup' of all elements of S dominated by U is called the dominion of U in S, denoted by $Dom_U S$. One can easily verify that

$$f \text{ is an epi iff } Dom_{Imf} T = Imf.$$

This motivates the following definition.

Definition 1.1. A subsemigroup (submonoid) U of a semigroup (monoid) is called *closed* if $Dom_U S = U$. We say that U is *absolutely closed* if it is closed in every oversemigroup (overmonoid).

Dominions and (absolute) closeness for posemigroups (pomonoids) are defined analogously.

2 Research Proposal 1

It follows from different results contained in [1, 2, 3] that a posemigroup (pomonoid) is absolutely closed if and only if its underlying semigroup (monoid) is such. The aim of this project will be to survey these articles and write a consolidated exposition on how the versions of absolute closeness coincide (in the categories of posemigroups and pomonoids). If time permits we shall also delve into the classes (categories) of pomonoids where the epimorphisms coincide with those of the underlying monoids.

3 Research Proposal 2

It follows from what is explained in the introduction that one may begin by considering two kinds of dominions in the categories of posemigroups and pomonoids: by considering or ignoring the orders. They are characterized by the following criteria

1. $d \in Dom_U S$ iff $d \otimes 1 = 1 \otimes d$, for monoids,
2. $d \in \hat{Dom}_U S$ iff $d \,\hat{\otimes}\, 1 = 1 \,\hat{\otimes}\, d$, for pomonoids,

where the tensor products are evaluated in the respective categories.

It turns out from [3] that $d \otimes 1 = 1 \otimes d$ if and only if $d \hat{\otimes} 1 = 1 \hat{\otimes} d$. The aim of this project is show that $x_1 \,\hat{\otimes}\, y_1 = x_2 \,\hat{\otimes}\, y_2$ does not imply, in general, that $x_1 \otimes y_1 = x_2 \otimes y_2$ in general.

As tensor products can be defined using first-order formulae, one may also consider using some automated theorem prover.

References

[1] S. Nasir: Zigzag Theorem for Partially Ordered Monoids, Comm. Algeb., 42:6, 2559–2583 (2014)

[2] S. Nasir and T. Lauri: Dominions, zigzags and epimorphismsfor partially ordered semigroups. ACUTM 18 (1), 81–91 (2014)

[3] S. Nasir: Epimorphisms, dominions and amalgamation in pomonoids, Semigroup Forum 90, 800–809 (2015)