Graduate Research Proposals

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1 Introduction and backgroud

Recall that a *semigroup* is a non-empty set S admitting an associative binary operation whereas a *monoid* is a semigroup with identity. If X is a non-empty set then the set $\mathcal{T}(X)$ of all transformations of X is a semigroup under the binary operation of usual composition of maps. Given a semigroup S, a homomorphism $\rho : S \to \mathcal{T}(X)$ is called a *representation* of S by $\mathcal{T}(X)$.

Definition 1.1. Let S be a monoid and A be a non-empty set. Then A is called a *right S-act* if there exists a right S-action $(a, s) \longmapsto as$, $a \in A$, $s \in S$ such that

1. for all $a \in A$, $s, t \in S$, $a(st) = (as)t$;
2. $a1 = a$, where 1 is the identity of S.

The notation A_S will mean that A is a right S-act. A *left S-act* ${}_SB$ is defined similarly.

If S is a monoid then every representation $\rho : S \to \mathcal{T}(X)$ turns X into a right S-act, where for all $x \in X$ and $s \in S$

$$xs = \rho_s(x).$$

Conversely, every right S-act X_S gives rise to a representation $s \mapsto \rho_s$ of S by $\mathcal{T}(X)$, where $\rho_s : X \to X$, $s \in S$, is defined by

$$\rho_s(x) = xs.$$

Definition 1.2. Let S be a subsemigroup of a semigroup T. Then S is said to have the *representation extension property* (REP) in T if for every representation $\rho : S \to \mathcal{T}(X)$ there exists a set $Y \supseteq X$ and a representation $\sigma : T \to \mathcal{T}(Y)$, given by $\sigma(t) = \sigma_t$, $t \in T$, such that

$$\sigma_s(x) = \rho_s(x)$$

for all $s \in S$ and $x \in X$. We say that S has representation extension property (REP) if it has (REP) in every oversemigroup.

Proposition 1.3. *A submonoid U of a monoid S has (REP) in S if for every right U-act X_U the map*

$$X \to X \otimes_U S \text{ given by } x \mapsto x \otimes 1,$$

is one-to-one.

Definition 1.4. Let S be a subsemigroup of a semigroup T. Then S is said to have the *right congruence extension property* (RCEP) in T if for every right congruence θ on S there exists a right congruence Θ on T such that

$$s_1 \Theta s_2 \text{ iff } s_1 \theta s_2,$$

for all $s_1, s_2 \in S$. We say that S has the congruence extension property (RCEP) if it has (RCEP) in every oversemigroup.

Definition 1.5. A right S-act A_S is called *flat* if for all left S-act embeddings

$$\phi : {}_S X \to {}_S Y$$

the induced map

$$1 \otimes \phi : A \otimes_S X \to A \otimes_U Y$$

is one-to-one. We call S *right absolutely flat* (RAF) if all its right acts are flat. *Left absolute flatness* is defined similarly. We call S *absolutely flat* if it is both right and left absolutely flat.

Remark 1.6. We have the following implications

$$(RAF) \Rightarrow (REP) \Rightarrow (RCEP).$$

Definition 1.7. A semigroup (monoid) *amalgam* is a list $\mathcal{A} = (U; S_1, S_2)$ of semigroups (monoids) with $S_1 \cap S_2 = U$. We say that $\mathcal{A}$ is embeddable if there exists a semigroup (monoid) T with S_1 and S_2 being embedded in T such that

1. the embedding monomorphisms $\phi_i : S_i \to T, 1 \leq i \leq 2$, agree on U,
2. and only U.

If only Condition (1) is satisfied then $\mathcal{A}$ is *weakly embeddable*.

Theorem 1.8. *If U is absolutely flat then the amalgam $(U; S_1, S_2)$ is embeddable for all S_1, S_2, i.e. U is an amalgamation base in the class of all semigroups.*

Remark 1.9. Inverse semigroups are absolutely flat, and hence amalgamation bases.

2 Research Project 1

Definition 2.1. A *posemigroup* $(S, \cdot, \leq)$, is a semigroup $(S, \cdot)$ endowed with a 'compatible' partial order $\leq$. *Pomonoids* are defined in a similar fashion.

Pomonoids are represented by S-posts, like monoids are by S-acts. Infact, (REP), (RCEP) and (RAF) may also be defined for pomonoids (posemigroups). The implications

$$(RAF) \Rightarrow (REP) \Rightarrow (RCEP).$$

also hold in the categories of all posemigroups and pomonoids. However, unlike monoids, the inverse pomonoids are not necessarily flat. Also, (RAF) may not imply the amalgamation property (as it does in the unordered case). The aim of this research project is to answer the following question.

Question 2.2. Are absolutely flat inverse pomonoids (posemigroups) amalgamation bases in the category of all pomonoids (posemigroups)?

3 Research Project 2

Definition 3.1. By a quantale we mean a posemigroup $(Q, \cdot, \leq)$, such that

1. $(Q, \leq)$ is a complete lattice,
2. $x \cdot (\vee M) = \vee(x \cdot M)$ and $(\vee M) \cdot x = \vee(M \cdot x)$ for all $x \in Q$ and all $M \subseteq Q$.

Quantales can be represented by modules of sup lattices. The aim of this project is answer the following (natural) questions.

Question 3.2. How to define (REP), (RCEP) and (RAF) for qualtales and how are they related with the amalgamation of quantales?

References

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