Studying the connection of Beth property
with the surjectivity of epimorphisms

Beth property (BP) is a logical property, whereas the surjectivity of epimorphisms (ES) is an algebraic property. We shall understand their connection.

By a logic $\mathcal{L}$ we mean an ordered quadruple ($F_{\mathcal{L}}$, $M_{\mathcal{L}}$, $mng_{\mathcal{L}}$, $\models_{\mathcal{L}}$), where,

1. $F_{\mathcal{L}}$ is the set of formulas of $\mathcal{L}$, which are recursively defined from a set P of atomic formulas by means of some fixed set of logical connectives $C_n(\mathcal{L})$
2. $M_{\mathcal{L}}$ is the class of models of $\mathcal{L}$.
3. $mng_{\mathcal{L}}$ is a family of functions

$$\left\{ mng_{\mathcal{L}}^{\mathfrak{M}} : \mathfrak{M} \in M_{\mathcal{L}} \right\},$$

   having domain $F_{\mathcal{L}}$. (For every $\mathfrak{M} \in M_{\mathcal{L}}$, the function $mng_{\mathcal{L}}^{\mathfrak{M}}$ provides us with a tool to interpret the formulas of $F_{\mathcal{L}}$ for the 'machine' $\mathfrak{M} \in M_{\mathcal{L}}$.)
4. $\models_{\mathcal{L}}$, called the validity relation, is a relation from $M_{\mathcal{L}}$ to $F_{\mathcal{L}}$. (For $\mathfrak{M} \in M_{\mathcal{L}}$ and $\varphi \in F_{\mathcal{L}}$ we interpret $\mathfrak{M} \models_{\mathcal{L}} \varphi$ as φ is valid in $\mathfrak{M}$).
5. For all $\varphi, \psi \in F_{\mathcal{L}}$ and $\mathfrak{M} \in M_{\mathcal{L}}$,

$$\text{if } \left[mng_{\mathcal{L}}^{\mathfrak{M}}\, \varphi = mng_{\mathcal{L}}^{\mathfrak{M}}\, \psi \text{ and } \mathfrak{M} \models_{\mathcal{L}} \varphi \right] \text{ then } \mathfrak{M} \models_{\mathcal{L}} \psi.$$

Example (Propositional Logic $\mathcal{L}_P$)

i. We have a set P of atomic formulas.

ii. The set of logical connectives of $\mathcal{L}_P$ is $\{\wedge,\ \neg\}$, $\wedge, \neg \notin P$.
   (Note that $(\psi \longrightarrow \phi) := \neg(\psi \wedge (\neg\phi))$ and $(\psi \vee \phi) := \neg(\neg\psi \wedge (\neg\phi))$.)

iii. $F_{\mathcal{L}_P}$ is the smallest set containing P such that

   (a) if $\psi \in F_{\mathcal{L}_P}$ then $\neg\psi \in F_{\mathcal{L}_P}$,
   (b) if $\psi, \phi \in F_{\mathcal{L}_P}$ then $\psi \wedge \phi \in F_{\mathcal{L}_P}$.

iv. We define the models of $\mathcal{L}_P$ as follows.
   For every non-empty set W and every function $\mu : P \longrightarrow \mathcal{P}(W)$ the pair $\langle W, \mu \rangle$ is a model of $\mathcal{L}_P$. (Indeed μ associates, to every atomic formula, a unique subset of W).

v. To define the meaning functions we first define a relation $\Vdash_\mu$ from W to P as follows.
   For all $w \in W$, we have

(a) if $\varphi \in P$ then $w \Vdash_\mu \varphi$ iff $w \in \mu(\varphi)$,

(b) if $\varphi = \neg\phi$ then then $w \Vdash_\mu \varphi$ iff $w \nVdash_\mu \mu(\phi)$,

(c) if $\varphi = \phi_1 \wedge \phi_2$ then $w \Vdash_\mu \varphi$ iff $w \Vdash_\mu \varphi_1$ and $w \Vdash_\mu \varphi_2$.

(d) Then we define $mng_{\mathcal{L}_P}^{\langle W,\mu\rangle} : F_{\mathcal{L}_P} \longrightarrow \mathcal{P}(W)$ by

$$mng_{\mathcal{L}_P}^{\langle W,\mu\rangle}(\varphi) = \{w \in W : w \Vdash_\mu \varphi\}.$$

vi. Finally, we define

$$\langle W, \mu\rangle \models_{\mathcal{L}_P} \varphi \text{ iff for all } w \in W,\ w \Vdash_\mu \varphi.$$

Algebras

(*These are non-technical descriptions*) We can think of $F_{\mathcal{L}_P}$ as free algebra over the signature $C_n(\mathcal{L})$. A logic $\mathcal{L}$ is called structural if

i. $mng_{\mathcal{L}}^{\mathfrak{M}}$ are homomorphisms from $F_{\mathcal{L}_P}$ to some algebra,

ii. all homomorphic images of $F_{\mathcal{L}_P}$ are models of $\mathcal{L}$.

Example Propositional logic (discussed above) is structural, with the class of Boolean Algebras as its associated class of algebras.

Implicit and explicit definitions

(*These are a non-technical descriptions*). We can imagine $x^2 - 2 = 0$ and $\sqrt{2}$; as implicit and explicit definitions, respectively, of the same real number. Every explicit definition is considered implicit. A logic $\mathcal{L}$ is said to have the Beth property if every relation (intuitively, concept) that is implicitly definable in $\mathcal{L}$ is explicitly definable.

BP vs ES

(*This is a non-technical description*) A structured logic has Beth property if and only if epimorphisms are surjective in its corresponding class of algebras.

Aims and scope

Writing an exposition based upon a thorough understanding of the above material. This study will also aim to explore, at the basic level, any links of ordered algebras with logic.

References

1. Eva Hoogland; Algebraic characterizations of various Beth definability properties. Studia Logica 65, 91–112 (2000)

2. István Németi. Beth definability is equivalent with surjectiveness of epis in general algebraic logic. Technical report, Math. Inst. Hungar. Acad. Sci., Budapest, (1984)

3. W. J. Blok and Don Pigozzi. Algebraizable Logics. Memoirs of the AMS, No 396