2-adic numbers, computing, and a problem for a 2-adic logarithm

We want to investigate the function $v(n)$, defined as the exponent of 2 in the prime factorization of the rational number

$A_n=2^1/1+2^2/2+2^3/3+\dots+2^n/n$.

For example, $A_3=2/1+4/2+8/3=4+8/3=20/3=2^2.3^{-1}.5$, so $v(3)=2$, and $A_4=A_3+16/4=32/3=2^5.3^{-1}$, so $v(4)=5$.

1. Show that $v(2^m-1)=2^m-m$;

2. Show that $v(n) \geq n-[\log n]$, where log is the base-2 logarithm and $[x]$ is the largest integer $\leq x$.

3*. Find a good upper bound for $v(n)$.

4*. Show that for n large enough, $v(n) \leq n+2[\log n]-2$, with equality if and only if n is a power of 2.

5^*. Show that $v(2^m)=2^m+2(m-2)$ for $m \geq 4$.

The statements in problems 1 and 2 are known, but any other result would be new. We want to investigate problems 3, 4 and 5 by computer experiments and by using a new expression for the numbers A_n.

These problems are related to 2-adic numbers and the 2-adic logarithm. Here, 2-adic numbers arize from a distance notion where two numbers are close if the exponent of 2 in the prime factorization of their difference is large. As a warm-up, we will investigate how 2-adic arithmetic is applied in compiler-writing. Then we will investigate the 2-adic logarithm and work on these problems. We can also think of generalizations to other primes.

The topic is suitable both for a bachelor thesis and for a master thesis.

Some background in number theory would help but is not required. For a master-level thesis some knowledge of number theory, linear algebra and algebraic structures (groups, rings, fields, completion) is probably needed.