International Journal of Theoretical and Applied Finance
Vol. 7, No. 2 (2004) 151–175

ANALYTICAL PRICING OF DOUBLE-BARRIER OPTIONS UNDER A DOUBLE-EXPONENTIAL JUMP DIFFUSION PROCESS: APPLICATIONS OF LAPLACE TRANSFORM

ARTUR SEPP

Institute of Mathematical Statistics, Faculty of Mathematics and Computer Science,
University of Tartu, J. Liivi 2, 50409 Tartu, Estonia
artursepp@hotmail.com
Web: www.hot.ee/seppar

Received 9 July 2003
Accepted 22 August 2003

We derive explicit formulas for pricing double (single) barrier and touch options with time-dependent rebates assuming that the asset price follows a double-exponential jump diffusion process. We also consider incorporating time-dependent volatility.

Assuming risk-neutrality, the value of a barrier option satisfies the generalized Black–Scholes equation with the appropriate boundary conditions. We take the Laplace transform of this equation in time and solve it explicitly. Option price and risk parameters are computed via the numerical inversion of the corresponding solution. Numerical examples reveal that the pricing formulas are easy to implement and they result in accurate prices and risk parameters.

Proposed formulas allow fast computing of smile-consistent prices of barrier and touch options.

Keywords: Jump diffusion processes; exponential jumps; volatility smile; option pricing; path-dependent options; double barrier options; double touch options; Laplace transform.

JEL classification code: G130

1. Introduction

The problem of pricing vanilla options consistent with the volatility smile has been attracting much attention in financial studies. Yet much has to be done to develop analytical methods for pricing exotic options under the smile. Various models have been proposed to explain the smile. One of the well-accepted models are jump diffusions or, more generally, Lévy processes. Merton [10] studied pricing of vanilla options on assets driven by jump diffusions with lognormally distributed price-jumps. Kou [4] proposed a jump diffusion with log-double-exponential jumps. It turns out that this jump diffusion has a few appealing features which allow analytical pricing of barrier and lookback options. Significant papers in this direction include those of Kou & Wang [5, 6], who worked out formulas for the distribution of the first exit time, single barriers, and floating strike lookback put options using memoryless property of the exponential distribution; and Lipton [7], who derived pricing formulas for single barrier options relying on fluctuation identities. Boyarchenko & Levendorskii [3] derived general pricing formulas for single barrier and touch options under a wide class of Lévy processes. The proposed approaches deal with the pricing problem in the Laplace domain.

Here, we develop an alternative approach to the pricing problem of single and double barrier options and derive explicit pricing formulas for double-barrier and double-touch options with time-dependent rebates under a double-exponential jump diffusion process. Pricing formulas for single barriers and touches are obtained via a simplification of general formulas for double barrier options. Our approach is less probabilistic and deals with the partial integro-differential equation (PIDE) supplied with the appropriate boundary conditions directly by taking the Laplace transform of this PIDE in time and solving it analytically.

2. Problem Formulation

The ubiquitous Black–Scholes [2] model assumes that the underlying follows a geometric Brownian motion with drift and diffusion parameters. In theory, the diffusion parameter, which is usually called volatility, is constant. In practice, all option markets exhibit a volatility smile phenomenon which means that options with different maturities and strikes have different implied Black–Scholes volatilities. As a result, accurate pricing and hedging of vanilla and exotic options is hard to achieve within the standard Black–Scholes model. For a comprehensive treatment of the pricing problem for double barriers and touches problem within the Black–Scholes framework and some of its extensions we refer to Lipton [9]. Lipton & McGhee [8] provide a good survey on pricing barrier options under various volatility models.

In our research we assume that the stochastic differential equation (SDE) governing the underlying price under the risk-neutral measure $\mathbb{Q}$ is given by

$$dS(t) = (r - d - \lambda\alpha)S(t)dt + (e^J - 1)S(t)dN(t) + \sigma S(t)dW(t), S(0) = S_0 \qquad (2.1)$$

where r is a risk-free (domestic) interest rate, d is a dividend (foreign interest) rate, σ is the volatility. $N(t)$ is a Poisson process with constant intensity λ, J is a random jump size with the probability density function (PDF) $\varpi(J)$. We set $\alpha = \mathbb{E}[e^J - 1]$ to make the discounted price process a martingale.

Although the choice of the PDF $\varpi(J)$ is rather unimportant for solving the pricing problem of vanilla options, this choice becomes essential for pricing exotic options. Here, we mainly consider the double-exponential PDF proposed by Kou [4]:

$$\varpi(J) = \varpi^-(J) + \varpi^+(J) = q^- \frac{1}{\eta^-} e^{\frac{1}{\eta^-}J} \mathbf{1}_{\{J<0\}} + q^+ \frac{1}{\eta^+} e^{-\frac{1}{\eta^+}J} \mathbf{1}_{\{J\geq 0\}} \qquad (2.2)$$

where $1 > \eta^+ > 0$ and $\eta^- > 0$ are means of positive and negative jump sizes, respectively; constants q^+ and q^- represent the probabilities of positive and negative jumps, respectively, $q^+, q^- \geq 0$, $q^+ + q^- = 1$. Requirement that $\eta^+ < 1$ is necessary to ensure that $\mathbb{E}[e^J] < \infty$ and $\mathbb{E}[S] < \infty$. We note that

$$\mathbb{E}[e^{\Phi J}] = \int_{-\infty}^{\infty} e^{\Phi J} \varpi(J) dJ = \frac{q^+}{1 - \Phi\eta^+} + \frac{q^-}{1 + \Phi\eta^-} \tag{2.3}$$

provided that $-\frac{1}{\eta^-} < \Phi < \frac{1}{\eta^+}$.

A simple calculation yields

$$\alpha = \mathbb{E}[e^J - 1] = \frac{q^+}{1 - \eta^+} + \frac{q^-}{1 + \eta^-} - 1\,. \tag{2.4}$$

Merton [10] proposed jump diffusion with lognormally distributed jumps. In general, the solution method developed here for a double-exponential jump diffusion cannot be applied for a jump diffusion with normally-distributed jumps. However, it follows from our subsequent analysis that the double-exponential distribution is more suitable for modeling jumps.

It is well-known that all jump diffusion models are incomplete. Throughout the paper we assume risk-neutrality. The value function $F(S, t)$ of a European-style option then satisfies the so-called generalized Black–Scholes PIDE (details can be found in Boyarchenko & Levendorskii [3]):

$$F_t + \frac{1}{2}\sigma^2 S^2 F_{SS} + (r - d - \lambda\alpha) S F_S - rF + \lambda \int_{-\infty}^{\infty} [F(Se^J) - F(S)]\varpi(J) dJ = 0\,,$$
$$F(S, T) = \max\{\varphi[S - K], 0\} \tag{2.5}$$

where subscripts indicate the partial derivatives, T is option maturity, K is strike, and binary variable $\varphi = +1$ for a call option and $\varphi = -1$ for a put option.

In a double-barrier knock-out option, the contract becomes worthless if either of the barriers is reached before the option expiry date. If neither of the barriers is hit, the double-barrier knock-out call (put) pays off $\max\{S - K, 0\}$ ($\max\{K - S, 0\}$) at the expiry date. In a double-barrier knock-in option, one of the barriers must be reached before the expiry date, otherwise the option becomes worthless.

Let S_u and S_d be the up and down barrier levels, respectively. The value of a double-barrier knock-out option denoted by $F^{DB}(S, t)$ satisfies PIDE (2.5) for $S_d < S < S_u$ subject to the following boundary conditions for $0 \leq t \leq T$:

$$F^{DB}(S, T - t) = \phi_u^*(T - t), S \geq S_u; F^{DB}(S, T - t) = \phi_d^*(T - t), S \leq S_d \tag{2.6}$$

where $\phi_u^*(t)$ and $\phi_d^*(t)$ are contract functions that determine payoffs if the corresponding up or down barrier is reached.

We usually have $\phi_u^*(t) = \phi_d^*(t) = 0$ for a standard double barrier option. Let R_u and R_d denote constant up and down rebates, respectively. If rebates are paid at the time when a barrier is hit, we have $\phi_u^*(t) = R_u$ and $\phi_d^*(t) = R_d$. If the payment is postponed until option maturity, we have $\phi_u^*(t) = e^{-rt} R_u$ and $\phi_d^*(t) = e^{-rt} R_d$.

We introduce new variables

$$t \to \tau = T - t, S \to x = \ln\frac{S}{K}, S_{u,d} \to x_{u,d} = \ln\frac{S_{u,d}}{K}, \phi_{u,d}(t) = \frac{\phi_{u,d}^*(t)}{K}\,,$$

and rewrite the pricing PIDE (2.5) for $F^{DB}(S,t) \to V^{DB}(x,\tau) = \frac{1}{K}F^{DB}(S,t)$ as

$$-V_\tau^{DB} + \frac{1}{2}\sigma^2 V_{xx}^{DB} + \mu V_x^{DB} - rV^{DB} + \lambda \int_{-\infty}^{\infty} [V^{DB}(x+J) - V^{DB}(x)]\varpi(J)dJ = 0\,,$$

$$\begin{aligned} &V^{DB}(x,0) = \max\{\varphi[e^x - 1], 0\}, x_d < x < x_u\,; \\ &V^{DB}(x,\tau) = \phi_u(\tau), x \geq x_u; V^{DB}(x,\tau) = \phi_d(\tau), x \leq x_d\,, \end{aligned} \tag{2.7}$$

where $\mu = r - d - \lambda m - \frac{1}{2}\sigma^2$.

A double-no-touch option pays off a constant amount of money if neither of the barriers is reached before the option expiry date. In opposite, a double-one-touch option pays off a fixed amount of money if either of the barriers is hit. The value of a double-touch option denoted by $F^{DT}(S,t)$ satisfies PIDE (2.5) subject to the boundary conditions (2.6) with the final payoff given by

$$F^{DT}(S,T) = K \tag{2.8}$$

where K is the contract payoff if neither of the barriers is hit. We usually have $K = 1$ without any rebates for double-no-touch options and $K = 0$ subject to rebates for double-one-touch options.

To allow pricing touch options with zero strike, we introduce variables $S \to y = \ln S$, $S_{u,d} \to y_{u,d} = \ln S_{u,d}$. The value of a double-one(no)-touch option $F^{DT}(S,t) \to V^{DT}(y,\tau)$ satisfies the following PIDE

$$-V_\tau^{DT} + \frac{1}{2}\sigma^2 V_{yy}^{DT} + \mu V_y^{DT} - rV^{DT} + \lambda \int_{-\infty}^{\infty} [V^{DT}(y+J) - V^{DT}(y)]\varpi(J)dJ = 0\,,$$

$$\begin{aligned} &V^{DT}(y,0) = K, y_d < y < y_u\,; \\ &V^{DT}(y,\tau) = \phi_u(\tau), y \geq y_u; V^{DT}(y,\tau) = \phi_d(\tau), y \leq y_d\,. \end{aligned} \tag{2.9}$$

For pricing knock-in barrier options, we can use the following useful and model-independent relationship

$$F_{\text{in}}(\varphi) = F_{\text{vanilla}}(\varphi) - F_{\text{out}}(\varphi) \tag{2.10}$$

where F_{in} is the value of double (single) barrier knock-in option, F_{out} is the value of double (single) barrier knock-out option and $F_{\text{vanilla}}(\varphi)$ is the value of vanilla option.

In our study we apply the Laplace transform to solve PIDEs (2.7) and (2.9). The Laplace transform is one of the classical methods for solving ODEs, PDEs, and integral equations. The idea behind the method is to transform the problem to a space where the solution is relatively easy to obtain. The corresponding solution is referred to as the solution in the Laplace domain. The original function can be retrieved either analytically via computing the Bromwich's integral or, in complicated cases, via a method of the numerical inversion.

The Laplace transform of $V(x,\tau)$ is defined by

$$U(x,p) := \mathcal{L}[V(x,\tau)] = \int_0^\infty V(x,\tau)e^{-p\tau}\,d\tau \tag{2.11}$$

where p is a transform variable with positive real part, $\Re p > 0$. To be specific, in subsequent analysis we assume that $p \in \mathbb{R}^+$. The standard rules yield

$$\mathcal{L}\left[\frac{\partial}{\partial\tau}V(x,\tau)\right] = pU(x,p) - V(x,0); \mathcal{L}\left[\frac{\partial^n}{\partial x^n}V(x,\tau)\right] = \frac{\partial^n}{\partial x^n}U(x,p)\,. \tag{2.12}$$

3. Pricing Under Constant Volatility

We start with a simpler constant volatility model and solve the corresponding problem for pricing vanilla and barrier options in the Laplace domain. We briefly review and generalize the framework proposed by Skachkov [14] and then extend it to jump diffusion processes.

We will use the following characteristic equation

$$\frac{1}{2}\sigma^2\xi^2 + \mu\xi - (r+p) = 0\,, \tag{3.1}$$

which has two real roots ξ_1 and ξ_2:

$$\xi_{1,2} = \xi_{+,-} = \frac{-\mu \pm \sqrt{\mu^2 + 2\sigma^2(r+p)}}{\sigma^2}\,, \tag{3.2}$$

such that $\xi_2 < 0 < \xi_1$.

3.1. *Vanilla options*

We set $\lambda = 0$ in PIDE (2.7) and omit the boundary conditions. Applying the Laplace transform to the corresponding PDE defined for $x \in (-\infty,\infty)$, we obtain the following ODE for $U(x,p) = \mathcal{L}[V(x,\tau)]$:

$$\frac{1}{2}\sigma^2 U_{xx} + \mu U_x - (r+p)U = -\max\{\varphi[e^x - 1], 0\}\,. \tag{3.3}$$

The solution to this ODE is specified by the following

Proposition 3.1. *In the Laplace domain, the value of a vanilla option is given by*

$$U(x,p) = \begin{cases} C_1 e^{\xi_1 x} + \dfrac{\varphi-1}{2}\left[\dfrac{e^x}{d+p} - \dfrac{1}{r+p}\right], & x < 0 \\ C_2 e^{\xi_2 x} + \dfrac{\varphi+1}{2}\left[\dfrac{e^x}{d+p} - \dfrac{1}{r+p}\right], & x \geq 0 \end{cases} \tag{3.4}$$

where

$$C_{1,2} = \frac{1}{\xi_1 - \xi_2}\left[\frac{\xi_{2,1}}{r+p} + \frac{1-\xi_{2,1}}{d+p}\right]\,. \tag{3.5}$$

Proof is given in Appendix A.2.

The option value in the time domain is found via the inversion of the Laplace transform:

$$F(S,F) = K\mathcal{L}^{(-1)}[U(x,p)]\,. \tag{3.6}$$

It is not difficult to find the original, which is exactly the Black–Scholes [2] formula, using a table of Laplace transforms. Pricing formula (3.4) can serve as a benchmark for testing the quality of the numerical inversion of the Laplace transform, which is vital for our subsequent developments.

3.2. *Barrier options*

Setting $\lambda = 0$ in PIDE (2.7) and applying the Laplace transform to the corresponding PDE, we obtain the boundary value problem for $U^{DB}(x,p) = \mathcal{L}[V^{DB}(x,\tau)]$:

$$\begin{gathered} \frac{1}{2}\sigma^2 U^{DB}_{xx} + \mu U^{DB}_x - (r+p)U^{DB} = -\max\{\varphi[e^x - 1], 0\}\,, \\ U^{DB}(x_u,p) = \overline{\phi_u}, U^{DB}(x_d,p) = \overline{\phi_d} \end{gathered} \tag{3.7}$$

where $\overline{\phi_u} = \mathcal{L}[\phi_u(\tau)]$ and $\overline{\phi_d} = \mathcal{L}[\phi_d(\tau)]$.

We note that $\overline{\phi_u} = 0$ and $\overline{\phi_d} = 0$ for a standard barrier option. For rebates of exponential form $\phi_{u,d}(\tau) = e^{-\rho_{u,d}\tau}R_{u,d}$, the corresponding Laplace transform is given by $\overline{\phi}_{u,d} = \frac{R_{u,d}}{\rho_{u,d}+p}$.

The solution to ODE (3.7) can be represented as

$$U^{DB}(x,p) = U^u(x,p) + U^b(x,p) \tag{3.8}$$

where U^u is a solution of unbounded ODE (3.7) (the value of a vanilla option given by Eq. (3.4)) and U^b is a solution of homogeneous equation corresponding to ODE (3.7) with boundary conditions

$$\begin{cases} U^b(x_d,p) = \overline{\phi_d} - U^u(x_d,p) \\ U^b(x_u,p) = \overline{\phi_u} - U^u(x_u,p) \end{cases}. \tag{3.9}$$

For $x_d \le x \le x_u$, the solution to $U^b(x_d,p)$ has a form

$$U^b(x,p) = C_3 e^{\xi_1 x} + C_4 e^{\xi_2 x}\,. \tag{3.10}$$

Plugging Eq. (3.10) into boundary conditions (3.9), we obtain a system of two equations to determine constants C_3 and C_4. Straightforward calculations yield the following

Proposition 3.2. *In the Laplace domain, the value of a double-barrier option is given by*

$$U^{DB}(x,p) = \begin{cases} (C_1 + C_3)e^{\xi_1 x} + C_4 e^{\xi_2 x} + \dfrac{\varphi - 1}{2}\left[\dfrac{e^x}{d+p} - \dfrac{1}{r+p}\right], & x < 0 \\ (C_2 + C_4)e^{\xi_2 x} + C_3 e^{\xi_1 x} + \dfrac{\varphi + 1}{2}\left[\dfrac{e^x}{d+p} - \dfrac{1}{r+p}\right], & x \ge 0 \end{cases} \tag{3.11}$$

where

$$\begin{aligned} C_{3,4} &= \pm\frac{1}{\chi}\left(-C_1 e^{\xi_1 x_d+\xi_{2,1}x_u} + C_2 e^{\xi_2 x_u+\xi_{2,1}x_d} - \frac{\varphi-1}{2}\left[\frac{e^{x_d+\xi_{2,1}x_u}}{d+p} - \frac{e^{\xi_{2,1}x_u}}{r+p}\right]\right. \\ &\quad \left. + \frac{\varphi+1}{2}\left[\frac{e^{x_u+\xi_{2,1}x_d}}{d+p} - \frac{e^{\xi_{2,1}x_d}}{r+p}\right] + e^{\xi_{2,1}x_u}\overline{\phi_d} - e^{\xi_{2,1}x_d}\overline{\phi_u}\right), \\ \chi &= e^{\xi_1 x_d+\xi_2 x_u} - e^{\xi_1 x_u+\xi_2 x_d}, \end{aligned} \tag{3.12}$$

and constants C_1, C_2 *are given by* (3.5).

The pricing problem for single barrier options can be solved via a simplification of the formula (3.11). In the presence of the down barrier, we let $x_u \to \infty$ and set $C_3 \equiv 0$. In the presence of the up barrier, we let $x_d \to -\infty$ and set $C_4 \equiv 0$.

The value of a double-barrier option is computed via the numerical inversion of the Laplace transform

$$F^{DB}(S,t) = K\mathcal{L}^{(-1)}[U^{DB}(x,p)]\,. \tag{3.13}$$

Now, we solve the pricing problem for double touch options. Setting $\lambda = 0$ in PIDE (2.9) and applying the Laplace transform to the corresponding PDE, we obtain the boundary value problem for $U^{DT}(y,p) = \mathcal{L}[V^{DT}(y,\tau)]$:

$$\begin{aligned} \frac{1}{2}\sigma^2 U^{DT}_{yy} + \mu U^{DT}_y - (r+p)U^{DT} &= -K\,, \\ U^{DT}(y_u,p) = \overline{\phi_u}, U^{DT}(y_d,p) &= \overline{\phi_d}\,. \end{aligned} \tag{3.14}$$

We represent $U^{DT}(y,p)$ by analogy with Eq. (3.8), where we take $U^u(y,p) = \frac{K}{r+p}$. We omit details to obtain:

Proposition 3.3. *In the Laplace domain, the value of a double-one*(*no*)*-touch option is given by*

$$U^{DT}(y,p) = C_1 e^{\xi_1 y} + C_2 e^{\xi_2 y} + \frac{K}{r+p} \tag{3.15}$$

where

$$\begin{aligned} C_{1,2} &= \pm\frac{1}{\chi}\left(K\frac{e^{\xi_{2,1}y_u} - e^{\xi_{2,1}y_d}}{r+p} + e^{\xi_{2,1}y_d}\overline{\phi_u} - e^{\xi_{2,1}y_u}\overline{\phi_d}\right), \\ \chi &= e^{\xi_1 y_u+\xi_2 y_d} - e^{\xi_1 y_d+\xi_2 y_u}\,. \end{aligned} \tag{3.16}$$

If a touch option has only the down barrier, we let $y_u \to \infty$ and set $C_2 \equiv 0$. If a touch option has only the up barrier, we let $y_d \to -\infty$ and set $C_1 \equiv 0$.

We have tested the numerical inversion of formulas (3.11) and (3.15) with some alternative pricing formulas obtained via the method of Green's function or Fourier series. The agreement between formulas is to the four decimals.

4. Pricing Vanillas under Double-Exponential Jumps

We solve the pricing problem for barrier options under a double-exponential jump diffusion in a similar manner: first we solve the unbounded problem of vanilla options, and next we consider the bounded problem of barrier options.

At the beginning, we consider the pricing problem of vanilla options under jump diffusions and state an original formula (in Laplace domain) for pricing vanilla options under a double-exponential jump diffusion. We will need the following

Lemma 4.1. *The equation*

$$\frac{1}{2}\sigma^2\psi^2 + \mu\psi - (r+p+\lambda) + \lambda\left[\frac{q^+}{1-\eta^+\psi} + \frac{q^-}{1+\eta^-\psi}\right] = 0 \tag{4.1}$$

has four real roots ψ_i, $i = 0, 1, 2, 3$, *such that*

$$-\infty < \psi_3 < -\frac{1}{\eta^-} < \psi_2 < 0 < \psi_1 < \frac{1}{\eta^+} < \psi_0 < \infty\,. \tag{4.2}$$

Proof is given in Appendix A.1. For computation purposes, it is better to represent Eq. (4.1) as the polynomial

$$\begin{aligned}
&\frac{1}{2}\sigma^2\eta^-\eta^+\psi^4 + \left(\mu\eta^-\eta^+ - \frac{1}{2}\sigma^2(\eta^- - \eta^+)\right)\psi^3 \\
&\quad - \left(\frac{1}{2}\sigma^2 + \mu(\eta^- - \eta^+) + (r+p+\lambda)\eta^-\eta^+\right)\psi^2 \\
&\quad + (-\mu + (r+p+\lambda)(\eta^- - \eta^+) - \lambda(q^+\eta^- - q^-\eta^+))\psi + (r+p) = 0\,.
\end{aligned} \tag{4.3}$$

Helpful algorithms for computing roots of this polynomial can be found, for example, in "Numerical recipes in C" [Press *et al* (1992)].

Taking the Laplace transform of the unbounded PIDE (2.7) and exchanging the order of integration, we obtain the following OIDE

$$\begin{aligned}
&\frac{1}{2}\sigma^2 U_{xx} + \mu U_x - (r+p+\lambda)U + \lambda\int_{-\infty}^{\infty}[U(x+J)]\varpi(J)dJ \\
&\qquad = -\max\{\varphi[e^x - 1], 0\}
\end{aligned} \tag{4.4}$$

defined for $x \in (-\infty, \infty)$.

The solution to the above equation is specified by the following:

Proposition 4.1. *In the Laplace domain, the value of a vanilla option under a double-exponential jump diffusion is given by*

$$U(x,p) = \begin{cases} C_0 e^{\psi_0 x} + C_1 e^{\psi_1 x} + \dfrac{\varphi - 1}{2}\left[\dfrac{e^x}{d+p} - \dfrac{1}{r+p}\right], & x < 0 \\[2ex] C_2 e^{\psi_2 x} + C_3 e^{\psi_3 x} + \dfrac{\varphi + 1}{2}\left[\dfrac{e^x}{d+p} - \dfrac{1}{r+p}\right], & x \geq 0 \end{cases} \tag{4.5}$$

where constants C_0, C_1, C_2, C_3 *are solution of the system*

$$\begin{pmatrix} 1 & 1 & -1 & -1 \\ \psi_0 & \psi_1 & -\psi_2 & -\psi_3 \\ \dfrac{1}{\psi_0\eta^- + 1} & \dfrac{1}{\psi_1\eta^- + 1} & -\dfrac{1}{\psi_2\eta^- + 1} & -\dfrac{1}{\psi_3\eta^- + 1} \\ \dfrac{1}{\psi_0\eta^+ - 1} & \dfrac{1}{\psi_1\eta^+ - 1} & -\dfrac{1}{\psi_2\eta^+ - 1} & -\dfrac{1}{\psi_3\eta^+ - 1} \end{pmatrix} \begin{pmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} \dfrac{1}{d+p} - \dfrac{1}{r+p} \\ \dfrac{1}{d+p} \\ \dfrac{1}{(d+p)(\eta^- + 1)} - \dfrac{1}{r+p} \\ \dfrac{1}{(d+p)(\eta^+ - 1)} + \dfrac{1}{r+p} \end{pmatrix}. \tag{4.6}$$

Proof is given in Appendix A.3.

Intuitively, the number of terms in the formula (4.5) is right. In the presence of positive jumps, it is clear that if $x < 0$ the point $x = 0$ can be crossed due to a positive jump, so it is necessary to account for such possibility. Let us consider the case when the jump amplitude becomes very small: $\eta^+ \to 0$ so that $\psi_0 \to \infty$. It is understandable that in this case the possibility is negligible and, accordingly, the corresponding term $C_0 e^{\psi_0 x}$ in formula (4.5) also becomes negligible. The same consideration applies for negative jumps in case $x > 0$.

The solution to system (4.6) can be computed using either the celebrated Cramer's rule or a purely numerical algorithm. We refer to "Numerical recipes in C" [Press *et al.* [11]] for helpful algorithms.

We have extensively tested numerical inversion of formula (4.5) with the semi-analytical formula derived by Sepp [13] via the Fourier transform of the option PIDE. We found that price differences between two formulas are less than 10^{-4} for most realistic values of model and option parameters. The advantage of considering the Laplace transform is that it allows pricing some path-dependent options in a similar setting. In general, the Fourier transform can be employed only for pricing vanillas.

Finally, we consider some limiting cases of double-exponential jumps. In the presence of only positive jumps we let $\eta^- \to 0$, $\psi_3 \to -\infty$ and set $C_3 = q^- \equiv 0$. As a result, the third equation in system (4.6) vanishes. In the presence of only negative jumps we let $\eta^+ \to 0$, $\psi_0 \to \infty$ and set $C_0 = q^+ \equiv 0$. As a result, the fourth equation in system (4.6) vanishes. When $\eta^+ \to 0$, $\psi_0 \to \infty$, $\eta^- \to 0$ $\psi_3 \to -\infty$, the formula (4.5) reduces to the formula (3.4) which we use for pricing vanillas under constant volatility. The same is true if $\lambda \to 0$.

5. Pricing Barriers Under Double-Exponential Jumps

Here, we consider the key contribution of our work: an analytical formula in the Laplace domain for pricing double-barrier options under a double-exponential jump diffusion process.

The pricing problem for double-barrier options in the presence of double-exponential jumps has the following representation in Laplace domain:

$$\begin{aligned}&\frac{1}{2}\sigma^2 U_{xx}^{DB} + \mu U_x^{DB} - (r+p+\lambda)U^{DB} + \lambda\int_{-\infty}^{\infty}[U^{DB}(x+J)]\varpi(J)dJ\\ &\qquad = -\max\{\varphi[e^x-1],0\}, x_d < x < x_d\,;\\ &U^{DB}(x,p) = \overline{\phi_d}, x \le x_d\,;\quad U^{DB}(x,p) = \overline{\phi_d}, x \ge x_u\,,\end{aligned} \tag{5.1}$$

where the jump size PDF $\varpi(J)$ is defined by Eq. (2.2).

The solution to Eq. (5.1) is specified by the following:

Proposition 5.1. *In the Laplace domain, the value of a double-barrier knock-out option under a double-exponential jump diffusion is given by formula*

$$U^{DB}(x,p) = \begin{cases} (C_0+C_4)e^{\psi_0 x} + (C_1+C_5)e^{\psi_1 x} + C_6 e^{\psi_2 x} + C_7 e^{\psi_3 x} \\ \qquad + \dfrac{\varphi-1}{2}\left[\dfrac{e^x}{d+p} - \dfrac{1}{r+p}\right], \quad x<0 \\ (C_2+C_6)e^{\psi_2 x} + (C_3+C_7)e^{\psi_3 x} + C_4 e^{\psi_0 x} + C_5 e^{\psi_1 x} \\ \qquad + \dfrac{\varphi+1}{2}\left[\dfrac{e^x}{d+p} - \dfrac{1}{r+p}\right], \quad x\ge 0 \end{cases} \tag{5.2}$$

where constants $C_j, j = 0,\ldots,3,$ *are solution of system* (4.6) *and constants* $C_j, j = 4,\ldots,7,$ *are solution of system*

$$\begin{pmatrix} \dfrac{e^{\psi_0 x_d}}{\psi_0\eta^- + 1} & \dfrac{e^{\psi_1 x_d}}{\psi_1\eta^- + 1} & \dfrac{e^{\psi_2 x_d}}{\psi_2\eta^- + 1} & \dfrac{e^{\psi_3 x_d}}{\psi_3\eta^- + 1} \\ e^{\psi_0 x_d} & e^{\psi_1 x_d} & e^{\psi_2 x_d} & e^{\psi_3 x_d} \\ e^{\psi_0 x_u} & e^{\psi_1 x_u} & e^{\psi_2 x_u} & e^{\psi_3 x_u} \\ \dfrac{e^{\psi_0 x_u}}{\psi_0\eta^+ - 1} & \dfrac{e^{\psi_1 x_u}}{\psi_1\eta^+ - 1} & \dfrac{e^{\psi_2 x_u}}{\psi_2\eta^+ - 1} & \dfrac{e^{\psi_3 x_u}}{\psi_3\eta^+ - 1} \end{pmatrix} \begin{pmatrix} C_4 \\ C_5 \\ C_6 \\ C_7 \end{pmatrix}$$

$$= \begin{pmatrix} -\dfrac{\varphi-1}{2}\left(\dfrac{e^{x_d}}{(d+p)(\eta^-+1)} - \dfrac{1}{r+p}\right) + \overline{\phi_d} - \dfrac{e^{\psi_0 x_d}}{\psi_0\eta^-+1}C_0 - \dfrac{e^{\psi_1 x_d}}{\psi_1\eta^-+1}C_1 \\ -\dfrac{\varphi-1}{2}\left(\dfrac{e^{x_d}}{(d+p)} - \dfrac{1}{r+p}\right) + \overline{\phi_d} - e^{\psi_0 x_d}C_0 - e^{\psi_1 x_d}C_1 \\ -\dfrac{\varphi+1}{2}\left(\dfrac{e^{x_u}}{(d+p)} - \dfrac{1}{r+p}\right) + \overline{\phi_u} - e^{\psi_2 x_u}C_2 - e^{\psi_3 x_u}C_3 \\ -\dfrac{\varphi+1}{2}\left(\dfrac{e^{x_u}}{(d+p)(\eta^+-1)} + \dfrac{1}{r+p}\right) - \overline{\phi_u} - \dfrac{e^{\psi_2 x_u}}{\psi_2\eta^+-1}C_2 - \dfrac{e^{\psi_3 x_u}}{\psi_3\eta^+-1}C_3 \end{pmatrix}. \tag{5.3}$$

Proof is given in Appendix A.4.

Intuitively, the number of terms in the formula (5.2) is right. A jump diffusion process can cross the down barrier due to a negative jump in both cases $x < 0$ and $x \geq 0$, so we have one term to account for such possibility and another term to account for crossing the barrier continuously. It is clear that these terms should vanish when $x \to \infty$, so we have to add in our formula the corresponding terms $e^{\psi_2 x}$ and $e^{\psi_3 x}$. It is also clear that the term $e^{\psi_3 x}$ is related to negative jumps and it must vanish when the jump size magnitude becomes negligible, that is, when $\eta^- \to 0$ and, accordingly, $\psi_3 \to -\infty$. The term $e^{\psi_2 x}$ is related to the regular diffusion and it is needed to account for crossing the down barrier continuously. The same considerations apply for the up barrier.

We also consider some limiting cases of the formula (5.2). For the single down barrier, we let $x_u \to \infty$ and set $C_4 = C_5 \equiv 0$. As a result, two first equations in system (5.3) vanish. For the single up barrier, we let $x_d \to -\infty$ and set $C_6 = C_7 \equiv 0$. As a result, two last equations in system (5.3) vanish.

In the presence of only positive jumps, we let $\eta^- \to 0$, $\psi_3 \to -\infty$ and set $C_3 = C_7 = q^- \equiv 0$. As a result, the first equation in system (5.3) vanishes. In presence of only negative jumps, we let $\eta^+ \to 0$, $\psi_0 \to \infty$ and set $C_0 = C_4 = q^+ \equiv 0$. As a result, the fourth equation in system (5.3) vanishes. When $\eta^+ \to 0$ and $\eta^- \to 0$, pricing formula (5.2) reduces to the formula (3.11) which we use for pricing barriers under constant volatility. The same is true when $\lambda \to 0$.

To derive the formula for the value of a double-barrier knock-in option denoted by $U^{DBKI}(x,p)$, we use relationship (2.10) as well as formulas (4.5) and (5.2). As a result, we obtain the following formula for $x \in (x_d, x_u)$:

$$U^{DBKI}(x,p) = -(C_4 e^{\psi_0 x} + C_5 e^{\psi_1 x} + C_6 e^{\psi_2 x} + C_7 e^{\psi_3 x})\,. \tag{5.4}$$

Finally, we solve the pricing problem for double-touch options. Taking the Laplace transform of Eq. (2.9), we obtain the following OIDE for $U^{DT}(y,p) = \mathcal{L}[V^{DT}(y,\tau)]$:

$$\begin{gathered}\frac{1}{2}\sigma^2 U^{DT}_{yy} + \mu U^{DT}_{y} - (r+p+\lambda)U^{DT} + \lambda \int_{-\infty}^{\infty} U^{DT}(y+J)\varpi(J)dJ = -K\,, \\ y_d < y < y_u; U^{DT}(y,p) = \bar{\phi}_u, y \geq y_u; U^{DT}(y,p) = \bar{\phi}_d, y \leq y_d\,. \end{gathered} \tag{5.5}$$

The solution is specified by the following:

Proposition 5.2. *In the Laplace domain, the value of a double-touch option under a double-exponential jump diffusion is given by formula*

$$U^{DT}(y,p) = C_4 e^{\psi_0 y} + C_5 e^{\psi_1 y} + C_6 e^{\psi_2 y} + C_7 e^{\psi_3 y} + \frac{K}{r+p} \tag{5.6}$$

where constants C_4, C_5, C_6, C_7 *are solution of system*

$$\begin{pmatrix} \frac{e^{\psi_0 y_d}}{\psi_0\eta^- + 1} & \frac{e^{\psi_1 y_d}}{\psi_1\eta^- + 1} & \frac{e^{\psi_2 y_d}}{\psi_2\eta^- + 1} & \frac{e^{\psi_3 y_d}}{\psi_3\eta^- + 1} \\ e^{\psi_0 y_d} & e^{\psi_1 y_d} & e^{\psi_2 y_d} & e^{\psi_3 y_d} \\ e^{\psi_0 y_u} & e^{\psi_1 y_u} & e^{\psi_2 y_u} & e^{\psi_3 y_u} \\ \frac{e^{\psi_0 y_u}}{\psi_0\eta^+ - 1} & \frac{e^{\psi_1 y_u}}{\psi_1\eta^+ - 1} & \frac{e^{\psi_2 y_u}}{\psi_2\eta^+ - 1} & \frac{e^{\psi_3 y_u}}{\psi_3\eta^+ - 1} \end{pmatrix} \begin{pmatrix} C_4 \\ C_5 \\ C_6 \\ C_7 \end{pmatrix} = \begin{pmatrix} -\frac{K}{r+p} + \overline{\phi_d} \\ -\frac{K}{r+p} + \overline{\phi_d} \\ -\frac{K}{r+p} + \overline{\phi_u} \\ \frac{K}{r+p} - \overline{\phi_u} \end{pmatrix}. \tag{5.7}$$

Proof is given in Appendix A.5.

We can treat limiting cases of the formula (5.6) by analogy with the aforementioned considerations for double-barrier options.

6. Pricing Under Jumps and Time-Dependent Volatility

All developed derivative markets exhibit the term structure of implied volatility which indicates that at-the-money (ATM) implied volatility depends on the expiry. Accordingly, the introduction of time-dependent volatility makes the pricing model more realistic. It is quite difficult to introduce stochastic volatility for analytical pricing of barrier options (see a survey by Lipton & McGhee [8]), but time-dependent volatility can be treated analytically in some cases.

We assume that under risk-neutral measure $\mathbb{Q}$ the asset price is driven by the following SDE:

$$\begin{cases} dS(t)/S(t) = (r - d - \widehat{\lambda} v(t)\alpha)dt + (e^J - 1)dN(t) + \sqrt{v(t)}dW(t), S(0) = S_0\,; \\ \qquad dv(t) = \kappa(v_\infty - v(t))dt, v(0) = v_0 \end{cases} \tag{6.1}$$

where v_0 is a spot variance, v_∞ is a long-term variance, and κ is a reversion speed to the long-term variance, $N(t)$ is a Poisson process with time-dependent intensity $\widehat{\lambda} v(t)$, $\widehat{\lambda}$ is a leverage coefficient.

We emphasize that in case of deterministic variance it is important to let the jump intensity be proportional with the variance. Integration of the variance dynamics yields

$$v(t) = v_\infty + (v_0 - v_\infty)e^{-\kappa t}\,. \tag{6.2}$$

Now, the pricing PIDE for the value of a barrier option becomes

$$F_t + \frac{1}{2}v(t)S^2 F_{SS} + (r - d - \widehat{\lambda}\alpha v(t))SF_S - rF + \widehat{\lambda}v(t)\mathbb{E}[F(Se^J) - F(S)] = 0\,,$$

$$F(S,T) = \max\{\varphi[S-K], 0\}$$

subject to boundary conditions (2.6).

For subsequent analysis, we need to remove terms which are not proportional with the variance. We introduce new variables

$$\tau = T - t, \nu = r - d, \widehat{S} = Se^{\nu\tau}, \widehat{S}_{u,d} = S_{u,d}e^{\nu\tau}, \widehat{v}(\tau) = v(T-\tau)\,,$$

$$\phi^*_{u,d}(T-t) = e^{-r\tau}\phi^\circ_{u,d}(\tau)\,,$$

and rewrite PIDE for $F(S,t) = e^{-r\tau}\Psi(\widehat{S},\tau)$ as

$$-\Psi_\tau + \frac{1}{2}\widehat{v}(\tau)\widehat{S}^2\Psi_{\widehat{S}\widehat{S}} - \widehat{\lambda}\alpha\widehat{v}(\tau)\widehat{S}\Psi_{\widehat{S}} + \widehat{\lambda}\widehat{v}(\tau)\mathbb{E}[\Psi(\widehat{S}e^J) - \Psi(\widehat{S})] = 0\,,$$

$$\Psi(\widehat{S},0) = \max\{\varphi[\widehat{S}-K], 0\}, \widehat{S}_d < \widehat{S} < \widehat{S}_u\,;$$

$$\Psi(\widehat{S},\tau) = \phi^\circ_u(\tau), \widehat{S} \geq \widehat{S}_u\,;$$

$$\Psi(\widehat{S},\tau) = \phi^\circ_d(\tau), \widehat{S} \leq \widehat{S}_d\,.$$

It is important to notice that now we have time-dependent barrier levels. It is known that we can remove time-dependence by introducing $\widehat{\tau} = \int_0^\tau \widehat{v}(s)ds$ [Lipton [9]]. We calculate the corresponding integral and introduce

$$\widehat{\tau} = v_\infty\tau + \frac{v_0 - v_\infty}{\kappa} - \left(\frac{v_0 - v_\infty}{\kappa}\right)e^{-\kappa\tau}\,, \quad \text{and} \quad \widehat{\Psi}(\widehat{S},\widehat{\tau}) = \Psi(\widehat{S},\tau)\,.$$

Using expression (6.2), we deduce that $\widehat{\Psi}(\widehat{S},\widehat{\tau})$ must satisfy

$$-\widehat{\Psi}_{\widehat{\tau}} + \frac{1}{2}\widehat{S}^2\widehat{\Psi}_{\widehat{S}\widehat{S}} - \widehat{\lambda}\alpha\widehat{S}\widehat{\Psi}_{\widehat{S}} + \widehat{\lambda}\mathbb{E}[\widehat{\Psi}(\widehat{S}e^J) - \widehat{\Psi}(\widehat{S})] = 0 \tag{6.3}$$

subject to the corresponding initial and boundary conditions.

This is PIDE with constant coefficients and we have already developed the method for solving it. Finally, we introduce $x = \ln\frac{\widehat{S}}{K}$, $x_{u,d}(\widehat{\tau}) = \ln\frac{\widehat{S}_{u,d}}{K}$, and rewrite the pricing PIDE for $\widehat{\Psi}(\widehat{S},\widehat{\tau}) \to \Xi(x,\widehat{\tau})$ as

$$\begin{aligned} &-\Xi_{\widehat{\tau}} + \frac{1}{2}\Xi_{xx} + \left(-\widehat{\lambda}\alpha - \frac{1}{2}\right)\Xi_x + \widehat{\lambda}\mathbb{E}[\Xi(x+J) - \Xi(x)] = 0\,,\\ &\Xi(x,0) = \max\{\varphi[e^x - 1], 0\}, x_d < x < x_u\,;\\ &\Xi(x,\widehat{\tau}) = \widehat{\phi}_u(\widehat{\tau}), x \geq x_u(\widehat{\tau}); \Xi(x,\widehat{\tau}) = \widehat{\phi}_d(\widehat{\tau}), x \leq x_d(\widehat{\tau})\,. \end{aligned} \tag{6.4}$$

The solution of the unbounded PIDE (6.4) in Laplace domain $\bar{\Xi}(x,p) = \mathcal{L}[\Xi(x,\widehat{\tau})]$ for pricing vanilla options under a double-exponential jump diffusion process with deterministic volatility and jump intensity is given by formula (4.5) where all expressions $d+p$ and $r+p$ have to be replaced by p and the corresponding variables have to be used.

Unfortunately, we can solve bounded PIDE (6.4) for pricing barrier and touch options only if $r-d=0$ because only in this case we have fixed (or flat) boundaries. In this case the general solution is given by formula (5.2) where all expressions $d+p$ and $r+p$ have to be replaced by p and the corresponding variables have to be used. In opposite case, we have moving boundaries which are rather difficult to handle analytically.

The value of a vanilla or barrier option is computed via numerical inversion of the Laplace transform (4.5) or (5.2) with corresponding variables:

$$F(S,t) = e^{-r\tau} K \mathcal{L}^{(-1)}[\bar{\Xi}(x,p)](\hat{\tau})\,.$$

We calibrated the pricing model with double-exponential jumps and time-dependent volatility to the whole DAX implied volatility surface of 5 July, 2002 and obtained the following estimates:

$v_0 = 0.32^2$, $v_\infty = 0.19^2$, $\kappa = 13.44$, $\hat{\lambda} = 25.37$, $\eta^+ = 0.13$, $\eta^- = 0.14$, $q^+ = 0.03$.

It follows that spot intensity equals $\lambda_0 = \hat{\lambda} v_0 = 2.57$ and long-term intensity equals $\lambda_\infty = \hat{\lambda} v_\infty = 0.9$.

The corresponding Black–Scholes implied volatility surface and differences between market and model implied volatilities are shown in Fig. 1.

It follows that the model can be calibrated to the whole volatility surface. However, the analytical pricing of barrier and touch options is possible only in case $r-d=0$ or in a special case when a barrier claim has moving barrier levels of the form $S_{d,u}(t) = S_{d,u} e^{-(r-d)t}$.

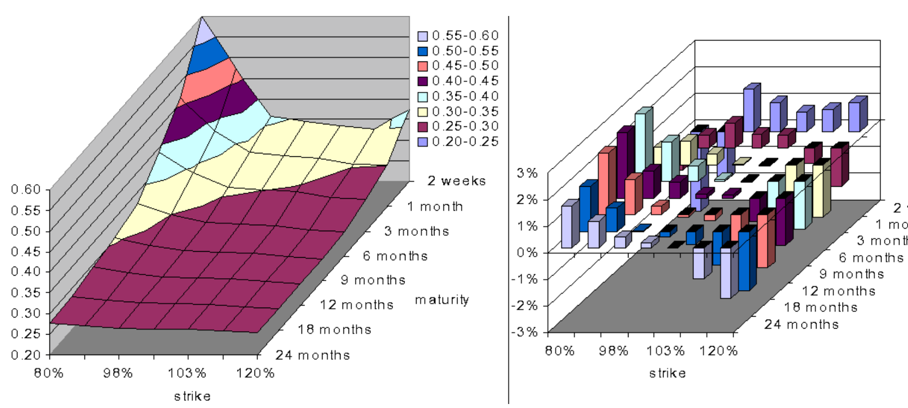

Fig. 1. The model implied volatility surface is shown on the left side. Differences between market and model implied volatility are shown on the right side. The double-exponential jump diffusion with time-dependent volatility is capable of producing both the volatility smile and the term structure of volatility.

7. Alternative Jump Size Distributions

Here, we consider some alternative PDFs for modeling jumps.

First, we note that our considerations can be applied to a mixture of positive and negative exponential jumps or the hyperexponential jump size PDF proposed by Lipton [7]. In general, by pricing vanilla options, each additional term in the mixture of exponential jumps will increase the order of characteristic polynomial (4.3) and the number of equations in system (4.6) by one.

7.1. *Laplacian jumps*

It is worthwhile to note that in exceptional case of double-exponential jumps we can obtain more convenient formulas for the pricing problem. In case of symmetric jumps, when $q^+ = q^- = \frac{1}{2}$ and $\eta = \eta^+ = \eta^-$, $1 > \eta > 0$, the jump size follows Laplace distributions. Replacing variable x in PIDE (2.7) for $z = x + \tau\mu$ and undertaking the previous analysis, we obtain fourth order characteristic polynomial that contains only terms with ψ^4 and ψ^2. The roots of this polynomial can be computed explicitly which simplifies the pricing problem. The solution of the pricing problem for vanillas is given by formula (4.5) where $\psi_j, j = 0, \ldots, 3$, are roots of the corresponding polynomial. We note that in this case it is easy to derive the explicit solution for this system. However, this change of the coordinate system complicates the pricing problem of barrier options.

7.2. *Analysis of lognormal jumps*

Merton [10] proposed the jump diffusion model where the logarithmic jump size is normally distributed with the PDF $\varpi(J) = \frac{1}{\sqrt{2\pi\delta^2}} e^{-\frac{(J-\nu)^2}{2\delta^2}}$.

The moment generating function is given by

$$\mathbb{E}[e^{\Phi J}] = e^{\nu\Phi + \frac{1}{2}\delta^2\Phi^2}\,.$$

By analogy, we obtain the following characteristic equation

$$\frac{1}{2}\sigma^2\psi^2 + \mu\psi - (r + p + \lambda) + \lambda e^{\nu\psi + \frac{1}{2}\delta^2\psi^2} = 0\,.$$

It is very difficult to analyze roots of this equation which is very important in our settings. Accordingly, the pricing problem in Laplace domain becomes rather involved. We note that Merton [10] derived a closed-form formula for pricing vanilla options under a jump diffusion with lognormally distributed price-jumps.

7.3. *Comparison of jump size distributions*

To compare jump diffusions with double-exponential jumps, negative exponential jumps, and normally distributed jumps, we calibrated three jump diffusion models to the DAX implied volatility of 5 July, 2002. We used options with the shortest maturity (2 weeks, $T = 0.04$) because the corresponding market implied volatility depicts very prominent smile feature. We obtained the following estimates for jump diffusion process with

(1) double-exponential jumps: $\sigma = 0.30$, $\lambda = 3.97$, $\eta^+ = 0.06$, $\eta^- = 0.10$, $q^+ = 0.15$;

(2) negative exponential jumps: $\sigma = 0.34$, $\lambda = 1.32$, $\eta^- = 0.14$;
(3) normally distributed jumps: $\sigma = 0.31$, $\lambda = 1.64$, $\nu = -0.13$, $\delta = 0.14$.

Based on these estimates it follows that, generally, jump models agree on jump size amplitudes. However, the double-exponential model has the largest jump intensity. Model implied volatilities are shown on the left side of Fig. 2.

We see that both the double-exponential and Merton's jump diffusions can successfully be calibrated to the market data. It is also clear that the jump diffusion with negative-exponential jumps cannot produce the smile feature of implied volatility. For calibrating to the whole volatility surface, we need to introduce the term structure of model parameters. For example, we can let σ depend on time. We considered one of such possibilities in Sec. 6.

It is also interesting to have a look at logarithmic jump size distributions implied by model parameters. The corresponding densities are shown on the right side of Fig. 2. We see that the Merton's jump model predicts a more significant negative impact of jumps. The double-exponential jump model predicts a larger number of smaller jumps than the Merton's model. The diffusion with negative-exponential jumps implies a more significant impact of negative jumps than the double-exponential jump diffusion.

We note that the evident advantage of the double-exponential jump diffusion is that it exhibits higher peaks which are inherent to the distributions of the asset returns. Another advantage of an exponential jump diffusion is that it is possible to obtain closed-form formulas for pricing of some exotic options.

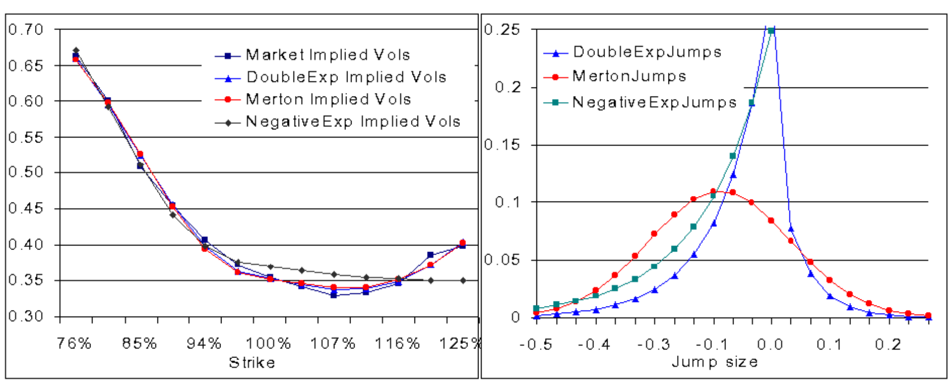

Fig. 2. The market and model Black–Scholes implied volatilities are shown on the left side. The implied jump size densities are shown on the right side. Both the double-exponential and the Merton's jump diffusions can be calibrated to market implied volatilities while producing quite different jump size magnitudes.

8. Numerical Inversion of the Laplace Transform

To invert the Laplace transform, we use the algorithm by Stehfest [15]. If $U(x,p)$ is the Laplace transform of $V(x,\tau)$, then the original can approximately be computed by

$$V(x,\tau) \approx \frac{\ln 2}{\tau}\sum_{j=1}^{N} Q_j U\left(x, j\frac{\ln 2}{\tau}\right) \tag{8.1}$$

where coefficients Q_j are given by

$$Q_j = (-1)^{N/2+j}\sum_{k=(j+1)/2}^{\min\{j,N/2\}} \frac{k^{N/2}(2k)!}{(N/2-k)!k!(k-1)!(j-k)!(2k-1)!}\,, \tag{8.2}$$

and N is an even number and k is computed using integer arithmetic.

The algorithm is very efficient and allows obtaining accuracy up to 8–10 significant digits. For more details on numerical inversion of the Laplace transform, see a survey by Abate *et al.* [1]. VBA code for the Stehfest algorithm can be found in Sepp & Skachkov [12]. For successful implementation of the algorithm, high precision arithmetic is a necessity. We use MSVS C++ compiler with long double types. We found that the choice of $N = 10 - 20$ results in satisfactory accuracy. The computational speed is very fast.

9. Numerical Results

As a numerical illustration, we shall present a few examples in Table 1 using the numerical inversion (NIL) of derived formulas. We shall also make a comparison with the Monte Carlo simulation. We use $K = 100$, $T = 1$, $t = 0$, $S_d = 80$, $S_u = 120$, $\sigma = 0.2$, $r = 0.05$, $d = 0.02$, $\eta^+ = 0.1$, $\eta^- = 0.1$, $q^+ = 0.5$, no rebates, and different levels of spot S_0 and jump intensity rate λ for calculating prices of vanilla call (VC), double-barrier knock-out call (DBOC), double-barrier knock-in put (DBIP) and double-no-touch (DNOT) option with strike $K = 1$.

All the computations are done on a Pentium 400 PC, 192 MB RAM. The Monte Carlo results are based on 20,000 simulation paths with 2000 time steps. The computation time is about 0.05–0.07 seconds for the numerical inversion of the Laplace transform of vanilla options and 0.1–0.15 seconds for barrier options (with N = 16 in the Stehfest algorithm). For the Monte-Carlo, the computation time is about 30–45 seconds. In addition to the point estimates, we also report the 95% confidence interval shown in parenthis.

Generally, the results obtained from the numerical inversion of Laplace transform and the Monte Carlo are comparable. We note that the price differences for vanilla options between the numerical inversion of the Laplace transform and the semi-analytical formula based on the Fourier transform [Sepp [13]] are less than 10^{-4} for most realistic values of option and model parameters. Reported results for

Table 1. Computations using the numerical inversion of Laplace transform.

S_0	λ	VC		DBOC	
		NIL	Monte–Carlo	NIL	Monte–Carlo
	0	4.3599	4.4316 (4.2999, 4.5633)	0.8283	0.8857 (0.8465, 0.9249)
90	3	8.2049	8.2118 (7.9523, 8.4712)	0.3668	0.4061 (0.3787, 0.4334)
	5	10.2478	10.3144 (9.9770, 10.6518)	0.2156	0.2528 (0.2307, 0.2749)
	0	9.2270	9.2668 (9.0757, 9.4580)	1.0730	1.1112 (1.0681, 1.1544)
100	3	13.3505	13.4221 (13.0885, 13.7557)	0.4743	0.4882 (0.4582, 0.5181)
	5	15.5462	14.8966 (14.4876, 15.3056)	0.2796	0.3084 (0.2844, 0.3323)
	0	15.9613	15.8738 (15.6262, 16.1214)	0.6957	0.7422 (0.7061, 0.7782)
110	3	19.7860	19.7516 (19.3493, 20.1539)	0.3309	0.3563 (0.3311, 0.3815)
	5	21.9267	22.5155 (22.0112, 23.0198)	0.2028	0.2197 (0.1996, 0.2398)
S_0	λ	DBIP		DNOT	
		NIL	Monte–Carlo	NIL	Monte–Carlo
	0	9.9134	9.8088 (9.6407, 9.9769)	0.2940	0.3023 (0.2961, 0.3084)
90	3	14.4758	14.2587 (14.0331, 14.4842)	0.1317	0.1380 (0.1334, 0.1427)
	5	16.7679	16.8082 (16.5588, 17.0577)	0.0780	0.0829 (0.0792, 0.0867)
	0	4.7698	4.8320 (4.7029, 4.9611)	0.3579	0.3668 (0.3574, 0.3702)
100	3	9.6648	9.7815 (9.5832, 9.9799)	0.1667	0.1750 (0.1699, 0.1801)
	5	12.1596	12.1392 (11.9127, 13.3657)	0.1000	0.1048 (0.1007, 0.1089)
	0	2.3404	2.3218 (2.2336, 2.4100)	0.2211	0.2272 (0.2215, 0.2328)
110	3	6.5537	6.4441 (6.2797, 6.6086)	0.1143	0.1174 (0.1131, 0.1218)
	5	8.8781	8.8210 (8.6225, 9.0194)	0.0720	0.0734 (0.0699, 0.0769)

the double barrier knock-out call option under the double-exponential jump diffusion have been verified by the anonymous referee from the IJTAF using a numerical solver for the PIDE. The agreement is also to the four decimals.

10. Empirical Investigation

It is a well-recognized fact that the pricing models can agree on prices of vanilla options but they may broadly disagree on prices of exotic options. To illustrate this issue, we calibrated the double-exponential jump diffusion model to the DAX implied volatility of 5 July, 2002. We used options with the middle-term maturity (6 months, $T \approx 0.46$) and obtained the following estimates:

$$\sigma = 0.182, \lambda = 1.459, \eta^+ = 0.01, \eta^- = 0.155, q^+ = 0.000\,.$$

The market (MIV) and model (DEJDIV) implied volatilities are reported in Table 2. It follows that the double-exponential jump diffusion can reproduce the market implied volatility in a very consistent manner.

Next we calculated prices (denoted by MV) of double-barrier knock-out calls (DBOC), double barrier knock-in puts (DBIP), double-no-touch (DNOT) with no rebates using pricing formula for constant volatility, where for every strike we used

Table 2. Computations using constant volatility formulas with market implied volatility and formulas incorporating double-exponential jumps.

	VOLS		DBOC		DBIP		DNOT	
strike	MIV	DEJDIV	MV	DEJD	MV	DEJD	MV	DEJD
3400	0.3667	0.3675	495.45	937.78	0.90	55.81	1774.56	2635.43
3600	0.3511	0.3499	429.52	786.52	7.09	74.74	2016.51	2790.46
3800	0.3327	0.3325	365.05	641.32	23.56	94.94	2303.37	2945.49
4000	0.3149	0.3158	298.04	505.04	54.45	115.78	2605.17	3100.51
4200	0.2999	0.3001	229.64	380.82	102.59	136.95	2894.93	3255.54
4400	0.2871	0.2859	165.85	271.92	170.00	158.33	3175.04	3410.56
4500	0.2781	0.2795	139.21	224.14	212.23	169.10	3348.21	3488.07
4600	0.2740	0.2735	111.84	181.18	258.53	179.91	3468.56	3565.59
4800	0.2631	0.2629	67.96	110.27	367.94	201.70	3745.97	3720.61
5000	0.2540	0.2540	35.90	59.18	497.25	223.75	4009.44	3875.64
5200	0.2463	0.2465	15.40	26.19	644.15	246.18	4260.80	4030.66
5400	0.2399	0.2403	4.58	8.19	805.18	269.28	4501.35	4185.69
5600	0.2356	0.2352	0.57	1.09	974.76	293.74	4720.99	4340.71

the corresponding market implied volatility. In addition, we calculated the same prices (denoted by DEJD) using the exponential jump diffusion with parameters reported earlier. As a basis we used: $S = 4483.03$, $T = 0.458$, $t = 0$, $S_d = 3200$, $S_u = 5800$, $r = 0.035$, $d = 0$ for different strikes. Our results are given in Table 2.

We see that the exponential jump model perfectly fits the prices of vanilla options and, at the same time, produces very different prices of barrier options. So a trader would be faced with the dilemma: which approach is to be used for pricing and hedging purposes, Black–Scholes pricing formulas with market implied volatilities or the double-exponential jump diffusion fitted to the market implied volatility. We cannot accept or reject either of the approaches based on the present observations. The final choice of an appropriate approach should be based on empirical results of applying these approaches for pricing and hedging barrier options in the real-life environment.

11. Conclusions

In this research we applied the Laplace transform for pricing barrier options. Our key contributions are formula (5.2) for pricing double (single) barrier and formula (5.6) for pricing double (single) touch options with time-dependent rebates under a double-exponential jump diffusion. The formulas allow pricing and hedging barrier options consistently with the volatility smiles and skews observed on the major derivative markets. In addition, if the differential rate is zero, it is possible to incorporate deterministic volatility for analytical pricing of barrier options.

Let us note that, although the formula (5.2) requires numerical computation of roots of the fourth order polynomial and solving two four by four matrix systems, it is still a semi-analytic solution, which allows computing option price and risk parameters accurately and quickly. The availability of a fast closed-form solution for barrier options allows using some liquid barrier options in the model calibration to market prices of liquid instruments. Thus, it is possible to make the pricing model be consistent with market prices of exotic options.

The market for barrier options is becoming very liquid. Accordingly, one needs a robust pricing formula which allows calculating option price and risk parameters quickly and accurately. It is also very important to take into account the volatility smile because barrier option prices are very sensitive to the market smiles and skews. The Laplace transform allows to achieve these very important goals.

Appendix

A.1. Proof of Lemma 4.1.

We show that Eq. (4.1) has four real roots such that relations (4.2) hold.

We consider Eq. (4.1) as a function $f(\psi)$. It follows that $f(\psi)$ is a convex function on interval $(-\frac{1}{\eta^-}, \frac{1}{\eta^+})$ with $f(0) = -(r+p) < 0$.

Since $\lim_{\psi \to (-\frac{1}{\eta^-})-} f(\psi) = +\infty$ and $\lim_{\psi \to (\frac{1}{\eta^+})-} f(\psi) = +\infty$, there is at least one root on the interval $(-\frac{1}{\eta^-}, 0)$ and another one on the interval $(0, \frac{1}{\eta^+})$.

Since $\lim_{\psi \to \frac{1}{\eta^+}+} f(\psi) = -\infty$ and $\lim_{\psi \to +\infty} f(\psi) = +\infty$, there is is at least one root on $(\frac{1}{\eta^+}, \infty)$.

By analogy, $\lim_{\psi \to -\infty} f(\psi) = -\infty$ and $\lim_{\psi \to (-\frac{1}{\eta^-})+} f(\psi) = +\infty$ imply that there is at least one root on the interval $(-\infty, -\frac{1}{\eta^-})$. Taking into the account that expression $(1-\eta^+\psi)(1+\eta^-\psi)f(\psi)$ represents a polynomial of the fourth order, it follows that the equation has exactly one root on each considered interval and the roots are real.

Finally, we consider expression $(1-\eta^+\psi)(1+\eta^-\psi)f(\psi)$ which yields polynomial (4.3).

A.2. Proof of Proposition 3.1.

Here, we prove formula (3.4) for pricing vanillas with constant volatility.

First we solve the homogeneous ODE corresponding to Eq. (3.3). We guess that the solution has a form $U(x,p) = e^{\psi x}$. Plugging it into Eq. (3.3), we get the corresponding characteristic Eq. (3.1).

Now we find a particular solution for non-homogeneous ODE corresponding to (3.3) which holds if

$$\varphi[e^x - 1] > 0\,.$$

We guess that the solution corresponding to homogeneous ODE has a form $U^h(x,p) = ae^x + b$. Plugging it into Eq. (3.3) and equating terms of the guessed solution, we obtain that for a call option

$$a = \frac{1}{d+p} \quad \text{and} \quad b = -\frac{1}{r+p}\,, \tag{A.1}$$

and for a put option

$$a = -\frac{1}{d+p} \quad \text{and} \quad b = \frac{1}{r+p}\,. \tag{A.2}$$

Thus, the general solution has the form

$$U^g(x,p) = \begin{cases} C_1 e^{\psi_1 x} + C_3 e^{\psi_2 x} + \dfrac{\varphi-1}{2}\left[\dfrac{e^x}{d+p} - \dfrac{1}{r+p}\right], & x \le 0 \\ C_4 e^{\psi_1 x} + C_2 e^{\psi_2 x} + \dfrac{\varphi+1}{2}\left[\dfrac{e^x}{d+p} - \dfrac{1}{r+p}\right], & x \ge 0\,. \end{cases} \tag{A.3}$$

To proceed further, we consider the limiting behavior of the values of call and put options in time domain. We have

$$F(S,t) \sim e^{-r(T-t)}\frac{\varphi-1}{2}[Se^{(r-d)(T-t)} - K]\,, \quad S \to 0\,; \tag{A.4}$$

and

$$F(S,t) \sim e^{-r(T-t)}\frac{\varphi+1}{2}[Se^{(r-d)(T-t)} - K]\,, \quad S \to \infty\,. \tag{A.5}$$

For normalized option values in the Laplace domain, formulas (A.4) and (A.5) correspond to the particular solution in formula (A.3). We require that the solution in Laplace domain should behave according to (A.4) and (A.5). This implies that $C_3 = 0$ and $C_4 = 0$.

To determine C_1 and C_2, we require that $U(x,p)$ and $U_x(x,p)$ are continuous at point $x = 0$. This leads to a system of two equations:

$$\begin{cases} U(x,p)|_{x=0-} = U(x,p)|_{x=0+} \\ U_x(x,p)|_{x=0-} = U_x(x,p)|_{x=0+} \end{cases}$$

Simple calculations yield the final expressions for C_1 and C_2 given by formula (3.5).

A.3. Proof of Proposition 4.1.

Here, we prove formula (4.5) for pricing vanilla options under a double-exponential jump diffusion. In the Laplace domain the option value satisfies OIDE (4.4).

A guess that the solution to the homogeneous OIDE (4.4) has the exponential form $U(x,p) = e^{\psi x}$ leads to the corresponding characteristic Eq. (4.1). We note that, although roots ψ_0 and ψ_3 do not satisfy the requirement of formula (2.3), the corresponding terms $e^{\psi_0 x}$ and $e^{\psi_3 x}$ are very important for solving OIDE (4.4). Below it will be clear that these terms also satisfy Eq. (4.1).

Now we introduce the operator

$$\Lambda(U) := \frac{1}{2}\sigma^2 U_{xx} + \mu U_x - (r+p+\lambda)U\,, \tag{A.6}$$

and define

$$U^-(x,p) = \sum_{i=0}^{1} C_i e^{\psi_i x} + \frac{\varphi - 1}{2}\left[\frac{e^x}{d+p} - \frac{1}{r+p}\right],$$
$$U^+(x,p) = \sum_{i=2}^{3} C_i e^{\psi_i x} + \frac{\varphi + 1}{2}\left[\frac{e^x}{d+p} - \frac{1}{r+p}\right]. \tag{A.7}$$

We make ansatz that $U(x,p)$ has the form

$$U(x,p) = U^-(x,p)\mathbf{1}_{\{x\le 0\}} + U^+(x,p)\mathbf{1}_{\{x\ge 0\}}. \tag{A.8}$$

We then determine constants C_j, $j = 0,1,2,3$ in such a way that OIDE (4.4) is satisfied. Two equations in system (4.6) arise from requiring the continuity of $U(x,p)$ and $U_x(x,p)$ at the point $x = 0$. Next we derive two additional equations.

We have two cases.

Firstly, if $x \le 0$. We consider the expectation

$$\begin{aligned}
&\int_{-\infty}^{\infty} U(x+J,p)\varpi(J)dJ \\
&= \int_{-\infty}^{0} U(x+J,p)\varpi^-(J)dJ + \int_{0}^{\infty} U(x+J,p)\varpi^+(J)dJ \\
&= \int_{-\infty}^{0} U^-(x+J,p)\varpi^-(J)dJ + \int_{0}^{-x} U^-(x+J,p)\varpi^+(J)dJ \\
&\quad + \int_{-x}^{\infty} U^+(x+J,p)\varpi^+(J)dJ \\
&= I_1 + I_2 + I_3 .
\end{aligned}$$

Straightforward calculations using PDF (2.2) yield

$$I_1 = q^- \sum_{i=0}^{1} \frac{C_i e^{\psi_i x}}{\eta^- \psi_i + 1} + q^- \frac{\varphi - 1}{2}\left[\frac{e^x}{(d+p)(1+\eta^-)} - \frac{1}{r+p}\right]$$

provided that $\psi_0 > -\frac{1}{\eta^-}$ and $\psi_1 > -\frac{1}{\eta^-}$, which is satisfied,

$$I_2 = -q^+ \sum_{i=0}^{1} \frac{C_i e^{\psi_i x}}{\eta^+ \psi_i - 1} + q^+ \sum_{i=0}^{1} \frac{C_i e^{\frac{1}{\eta^+}x}}{\eta^+ \psi_i - 1} + q^+ \frac{\varphi - 1}{2}\left[\frac{e^{\frac{1}{\eta^+}x} - e^x}{(d+p)(\eta^+ - 1)} - \frac{1 - e^{\frac{1}{\eta^+}x}}{r+p}\right]$$

and finally

$$I_3 = -q^+ \sum_{i=2}^{3} \frac{C_i e^{\frac{1}{\eta^+}x}}{\eta^+ \psi_i - 1} - q^+ \frac{\varphi + 1}{2}\left[\frac{e^{\frac{1}{\eta^+}x}}{(d+p)(\eta^+ - 1)} + \frac{e^{\frac{1}{\eta^+}x}}{r+p}\right]$$

provided that $\psi_2 < \frac{1}{\eta^+}$ and $\psi_3 < \frac{1}{\eta^+}$, which is satisfied.

We make terms with $e^{\frac{1}{\eta^+}x}$ vanish which yields the fourth equation in system (4.6).

Next we consider the final expression

$$\Lambda(U^-) + \lambda(I_1 + I_2 + I_3)\,. \tag{A.9}$$

It follows that terms with C_0 and C_1 vanish due to the characteristic Eq. (4.1). We then verify that the remaining terms in (A.9) represent the right side of OIDE (4.4), which is indeed the case.

Secondly, if $x > 0$. We consider the expectation

$$\begin{aligned}
&\int_{-\infty}^{\infty} U(x+J,p)\varpi(J)dJ \\
&\quad = \int_{-\infty}^{0} U(x+J,p)\varpi^-(J)dJ + \int_{0}^{\infty} U(x+J,p)\varpi^+(J)dJ \\
&\quad = \int_{-\infty}^{-x} U^-(x+J,p)\varpi^-(J)dJ + \int_{-x}^{0} U^+(x+J,p)\varpi^-(J)dJ \\
&\qquad + \int_{0}^{\infty} U^+(x+J,p)\varpi^+(J)dJ\,.
\end{aligned}$$

We calculate these integrals using PDF (2.2) and apply the same considerations to make vanish the terms with $e^{-\frac{1}{\eta^-}x}$ and those with C_2, C_3. As a result, we derive the third equation in system (4.6).

As a result, the representation (A.8) is valid and we obtain the system (4.6) for determination of constants C_0, C_1, C_2, and C_3.

A.4. Proof of Proposition 5.1.

Here, we prove formula (5.2) for pricing double-barrier options under a double-exponential jump diffusion.

We present the solution of the value of a double-barrier option as a superposition of the unbounded solution $U^u(x,p)$ for the vanilla option given by formula (4.5) and the bounded solution denoted by $U^b(x,p)$ as

$$U^{DB}(x,p) = U^u(x,p) + U^b(x,p) \tag{A.10}$$

where

$$U^b(x,p) = \begin{cases} \overline{\phi_d} - U^u(x,p)\,, & x \le x_d \\ \displaystyle\sum_{i=4}^{7} C_i e^{\psi_{i-4}x}\,, & x_d < x < x_u \\ \overline{\phi_u} - U^u(x,p)\,, & x \ge x_u \end{cases}\,. \tag{A.11}$$

We determine coefficients $C_i, i = 4, \ldots, 7$, in such a way that boundary conditions are matched and $U^b(x,p)$ satisfies the homogeneous OIDE (5.1).

First we consider the expectation

$$\int_{-\infty}^{\infty} U^b(x+J,p)\varpi(J)dJ$$

$$= \int_{-\infty}^{0} U^b(x+J,p)\varpi^-(J)dJ + \int_{0}^{\infty} U^b(x+J,p)\varpi^+(J)dJ$$

$$= \int_{-\infty}^{x_d-x} [\overline{\phi_d} - U^u(x+J,p)]\varpi^-(J)dJ + \int_{x_d-x}^{0} \sum_{i=4}^{7} C_i e^{\psi_{i-4}(x+J)}\varpi^-(J)dJ$$

$$+ \int_{0}^{x_u-x} \sum_{i=4}^{7} C_i e^{\psi_{i-4}(x+J)}\varpi^+(J)dJ + \int_{x_u-x}^{\infty} [\overline{\phi_u} - U^u(x+J,p)]\varpi^+(J)dJ\,.$$

Next we calculate above integrals using PDF (2.2). We note that $U^u(x,p)$ for $x \le x_d$ is given by formula (4.5) for a vanilla option with $x \le 0$ and $U^u(x,p)$ for $x \ge x_u$ is given by formula (4.5) with $x > 0$. The final expression contains terms with $e^{\frac{1}{\eta^-}(x_d-x)}$ and $e^{-\frac{1}{\eta^+}(x_u-x)}$. We make them vanish by equating sums of corresponding coefficients to zero. As a result, we obtain the first and fourth equation in system (5.3), respectively.

Next we consider the sum of $\Lambda(U^b)$, where operator Λ is defined in (A.6), with remaining terms of the above integrals. It then follows that all remaining terms with $C_i, i = 4,\ldots,7$, vanish due to characteristic Eq. (4.1).

The second and third equation in matrix (5.3) arise from the down and up boundary condition of the barrier option, respectively. Thus, representation (A.10) is valid and we obtained system (5.3) for determination of constants $C_i, i = 4,\ldots,7$.

A.5. Proof of Proposition 5.2.

Here, we outline the proof of formula (5.6) for pricing double-touch options under a double-exponential jump diffusion. We consider the bounded problem for $U^{DT}(y,p)$ which has the representation given by formula (A.10) with $U^u(y,p) = \frac{K}{r+p}$ and $x \equiv y$. Next we apply the same consideration as in Appendix A.4. and derive system (5.7).

Acknowledgment

I am grateful to Raul Kangro, Igor Skachkov, Alan Lewis and Kalev Pärna for useful suggestions and comments. I also thank the anonymous referee from the IJTAF for valuable advice, which has lead to the improvement of the paper. This research was partially supported by Estonian Science Foundation grant no 5022.

References

[1] J. Abate, G. Choudhury and W. Whitt, An introduction to numerical inversion and its application to probability models, in *Computational Probability*, ed. W. Grassman (Kluger, Boston, 1999), pp. 257–323.

[2] F. Black and M. Scholes, The pricing of options and corporate liabilities, *Journal of Political Economy* **81** (1973) 637–659.

[3] S. Boyarchenko and S. Levendorskii, Barrier options and touch-and-out options under regular Lévy processes of exponential type, *Annals of Applied Probability* **12**(4) (2002) 1261–1298.

[4] S. Kou, A jump diffusion model for option pricing, *Management Science* **48** (2002) 1086–1101.

[5] S. Kou and H. Wang, First passage times of a jump diffusion process, *Advances in Applied Probability* **35** (2003) 504–531.

[6] S. Kou and H. Wang, *Option Pricing Under a Double Exponential Jump Diffusion Model*, forthcoming in *Management Science* (2004).

[7] A. Lipton, Assets with Jumps, *Risk* (September 2002) 149–153.

[8] A. Lipton and W. McGhee, Universal Barriers, *Risk* (May 2002) 81–85.

[9] A. Lipton, *Mathematical Methods for Foreign Exchange: A Financial Engineer's Approach* (World Scientific Publishing Co., Inc., 2001).

[10] R. Merton, Option pricing when underlying stock returns are discontinuous, *Journal of Financial Economics* **3** (1976) 125–144.

[11] W. Press, B. Flannery, S. Teukolsky and W. Vetterling, Numerical recipes in C: The Art of Scientific Computing (Cambridge Univ. Press, 1992), available at www.nr.com.

[12] A. Sepp and I. Skachkov, Option Pricing with Jumps, *Wilmott magazine* (November 2003) 50–58.

[13] A. Sepp, *Pricing European-Style Options under Jump Diffusion Processes with Stochastic Volatility: Applications of Fourier Transform*, University of Tartu, Preprint (2003), available at www.hot.ee/seppar.

[14] I. Skachkov, *Black–Scholes Equation in Laplace Transform Domain*, Wilmott.com, Technical Article (2002), available at www.wilmott.com.

[15] H. Stehfest, Numerical inversion of Laplace transforms, *Communications of the Association for Computing Machinery* **13**(1) (1970) 47–49.