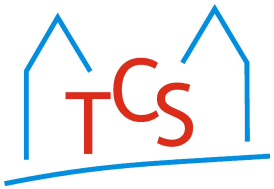
On String Compression by Context-free Grammars

Rüdiger Reischuk

joint work with Jan Arpe

presented at Data Compression Conference, Utah 2006



Institut für Theoretische Informatik
Universität zu Lübeck

2. Lübeck-Tartu Workshop
September 26, 2006

Research at ITCS

- algorithmic complexity

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  - design efficient algorithms: randomized, parallel, online

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  - authentication and privacy issues in networks
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   - simple computational model: formal grammars

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main concern in the following: find a **small grammar G_x for x**!
coding of G_x can be done by standard methods

Example 1

greedy strategy:

iteratively replace longest subwords that appear more than once by a nonterminal

Example 1

Input

$x = \texttt{fischers} \sqcup \texttt{fritz} \sqcup \texttt{fischt} \sqcup \texttt{frische} \sqcup \texttt{fische}$

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Compression Rate

$$|x| = 36 \quad \rightsquigarrow \quad |G_x| = 26$$

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Minimum Grammar Compression

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Optimization Problem
Minimum Grammar Compression

k-MGC bounded alphabet size

- input: $x \in \Sigma^*$, $|\Sigma| \leq k$
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Optimization Problem
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2-MGC binary alphabet

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  – nonconstant lower bound for MGC?

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  - in particular, $m^*(x) \leftrightarrow m^*(\varphi(x))$?

From G_x to $G_{\varphi(x)}$

Example 2

string	*transformation*
$x = \mathtt{abcaabc}$	$\varphi(\mathtt{a}) = 1001,\ \varphi(\mathtt{b}) = 0010$
	$\varphi(\mathtt{c}) = 0100$

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G_x :

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$$|G_{\varphi(x)}| = |G_x| + |G_\varphi| = 6 + 11 = 17$$

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Theorem

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$$\mathbf{m^*(\varphi(x)) \ \leq \ m^*(x) + m^*(\varphi(\Gamma))}$$

From $G_{\varphi(x)}$ to G_x

Example 3

$x = \mathtt{abcaabc}$ $\qquad$ $\varphi(\mathtt{a}) = 1001,\ \varphi(\mathtt{b}) = 0010$
$\varphi(\mathtt{c}) = 0100$

$G_{\varphi(x)}$:

S	$\longrightarrow$	PAP		A	$\longrightarrow$	1T1
P	$\longrightarrow$	ABC		B	$\longrightarrow$	T10
				C	$\longrightarrow$	01T
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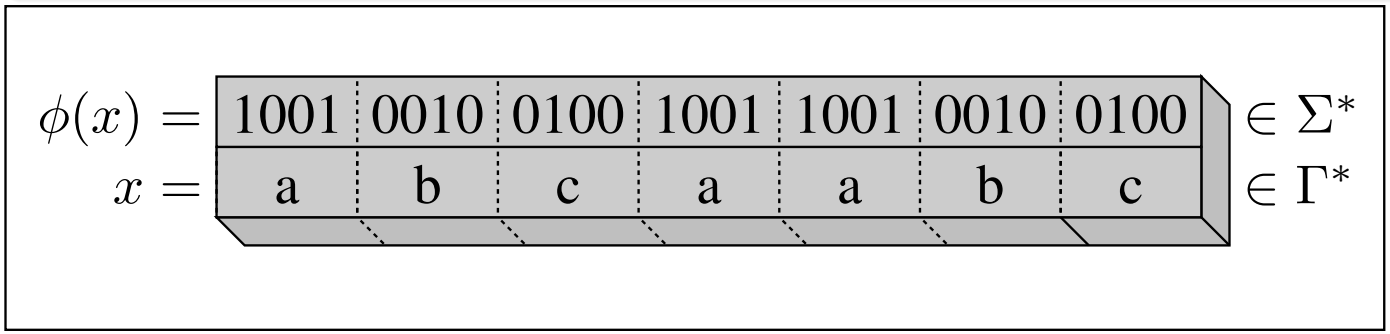

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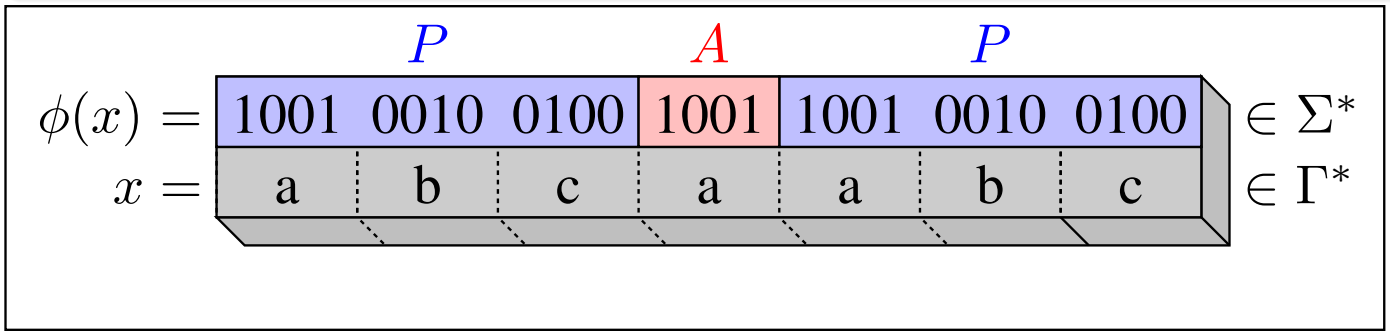

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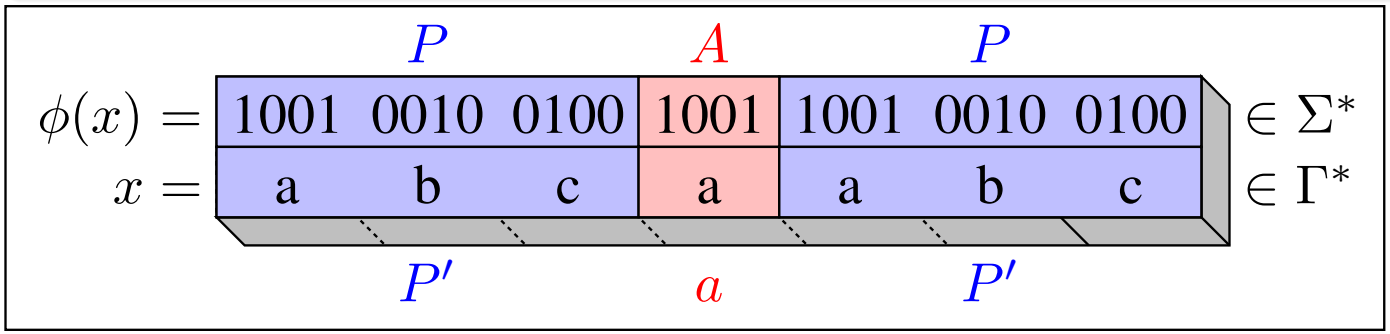

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? always true ?

$$|G_x| \leq |G_{\varphi(x)}| - m^*(\varphi(\Gamma)) \quad ?$$

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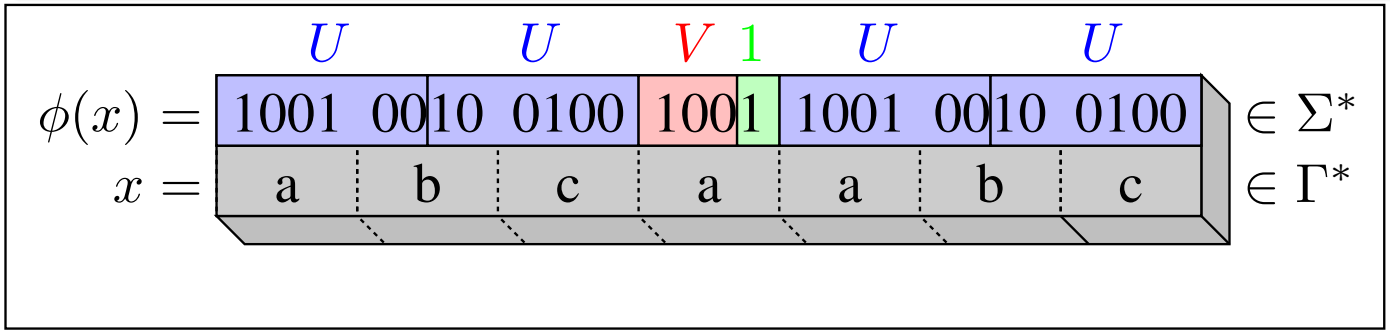

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From $G_{\varphi(x)}$ to G_x: lower bound

Small alphabets may compress much better

Theorem

For every ℓ there exists an alphabet Γ of size 2^ℓ,
a binary blockcode $\varphi : \Gamma^* \longrightarrow \{0,1\}^*$ and
a string $x \in \Gamma^*$ of size about $2^{2\ell}$ such that

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$$m^*(\varphi(x)) \ \leq \ O(2^\ell), \qquad m^*(\varphi(\Gamma)) \ \leq \ O(2^\ell), \qquad m^*(x) \ \geq \ \Omega(\ell \cdot 2^\ell)$$

From $G_{\varphi(x)}$ to G_x: upper bound

This loss of compression is the worst for ℓ-blockcodes

Theorem

If φ is an ℓ-blockcode and $G_{\varphi(x)}$ a grammar for a coded string $\varphi(x)$

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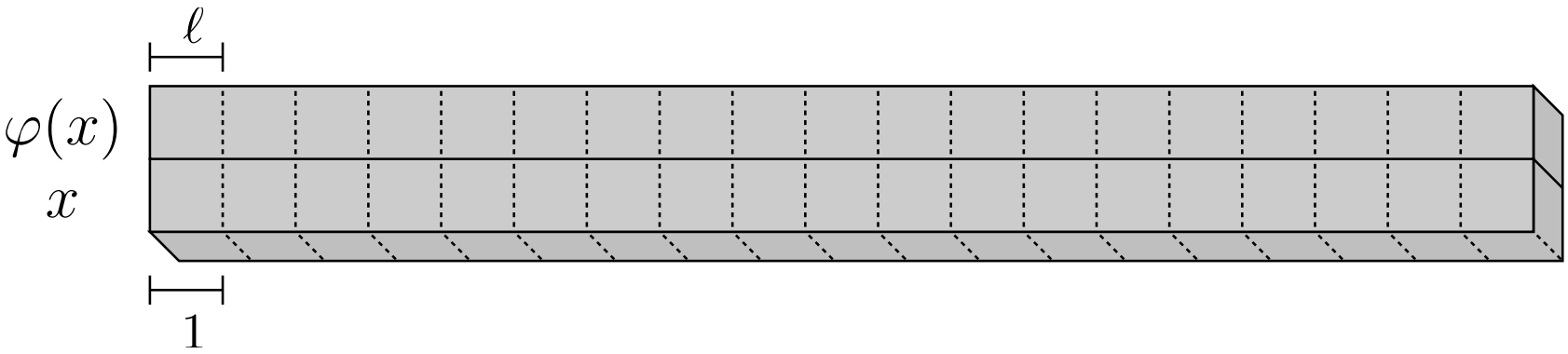
From $G_{\varphi(x)}$ to G_x: upper bound - idea of proof

in $G_{\varphi(x)}$: $A \to \alpha$



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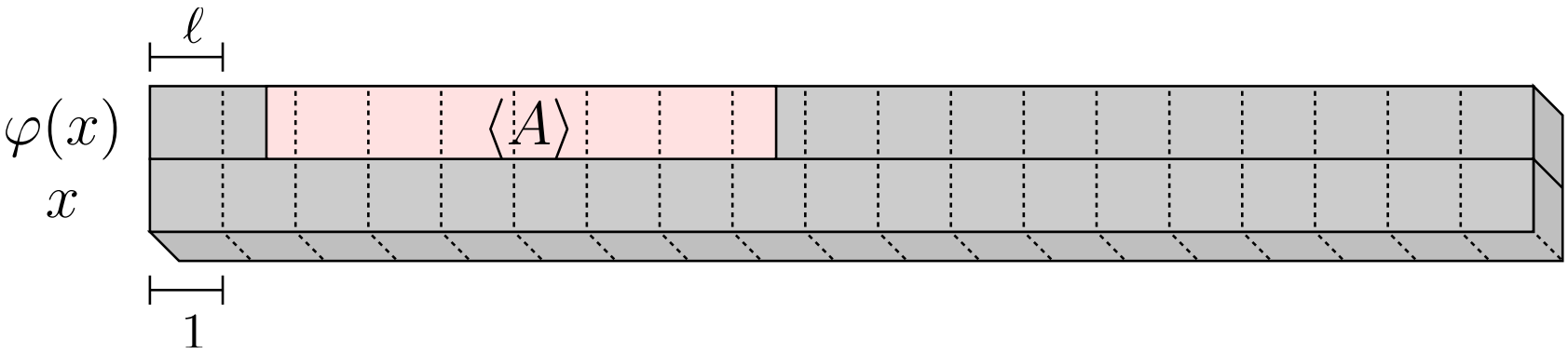

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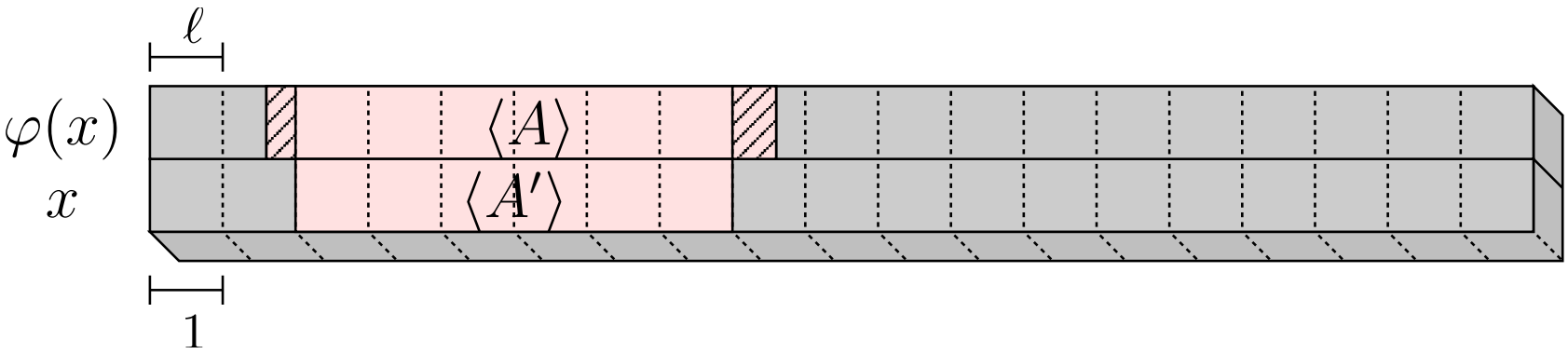

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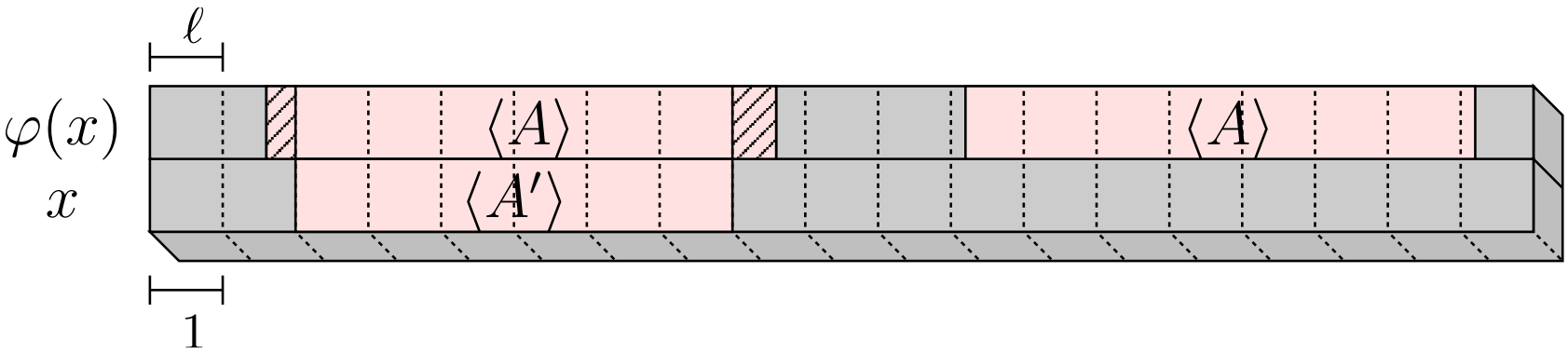

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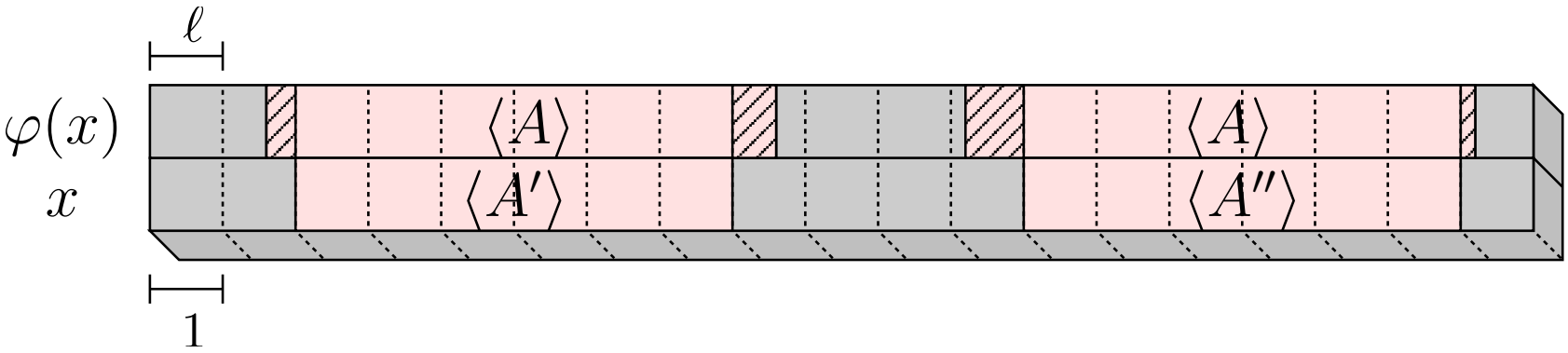

in G_x: $A' \to \alpha'$ $A'' \to \alpha''$ ... max. ℓ copies

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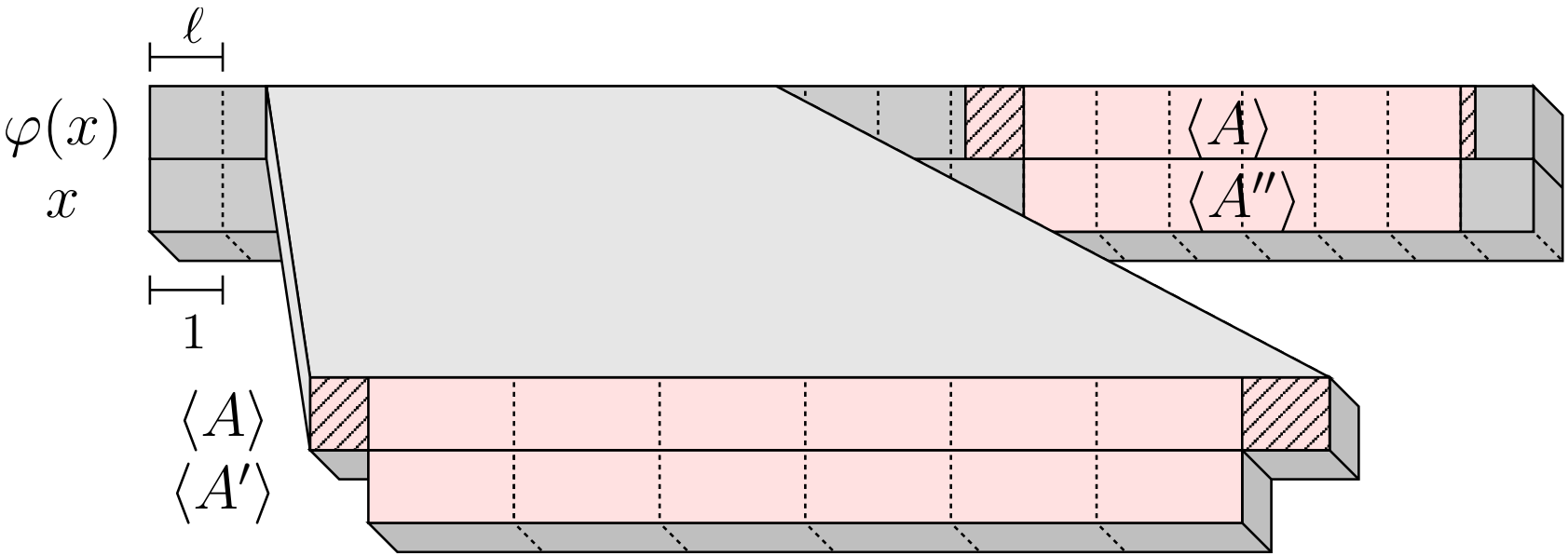

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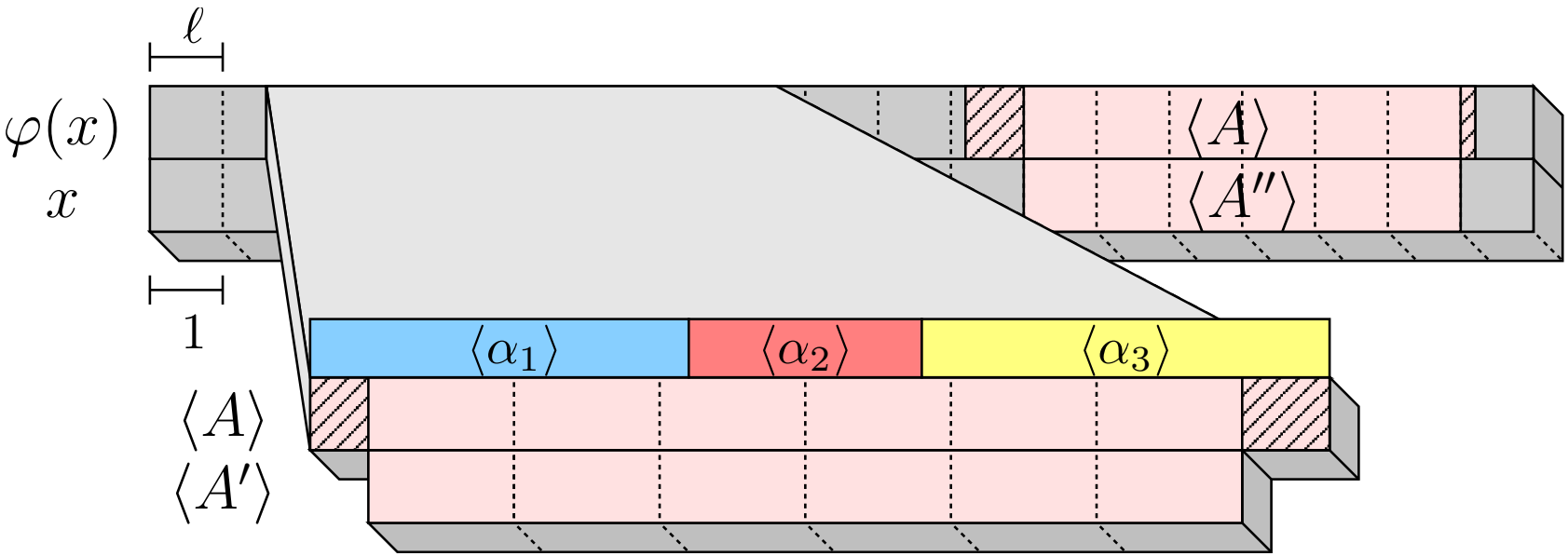

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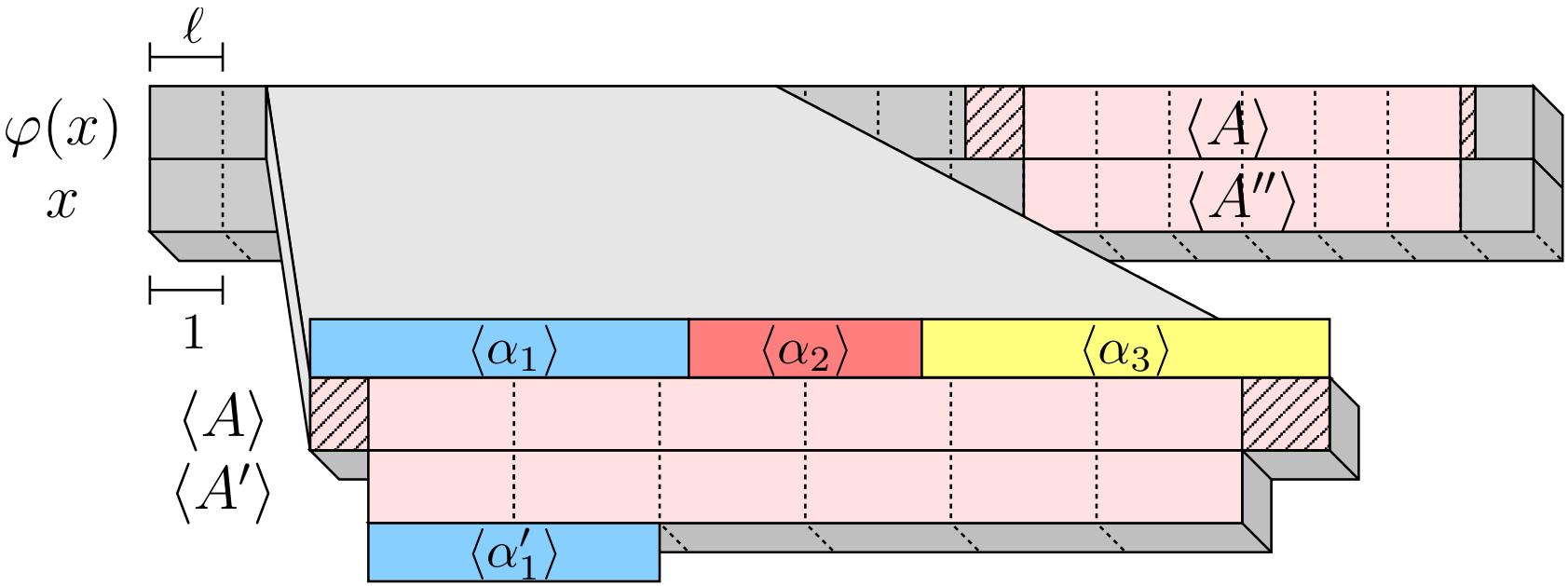

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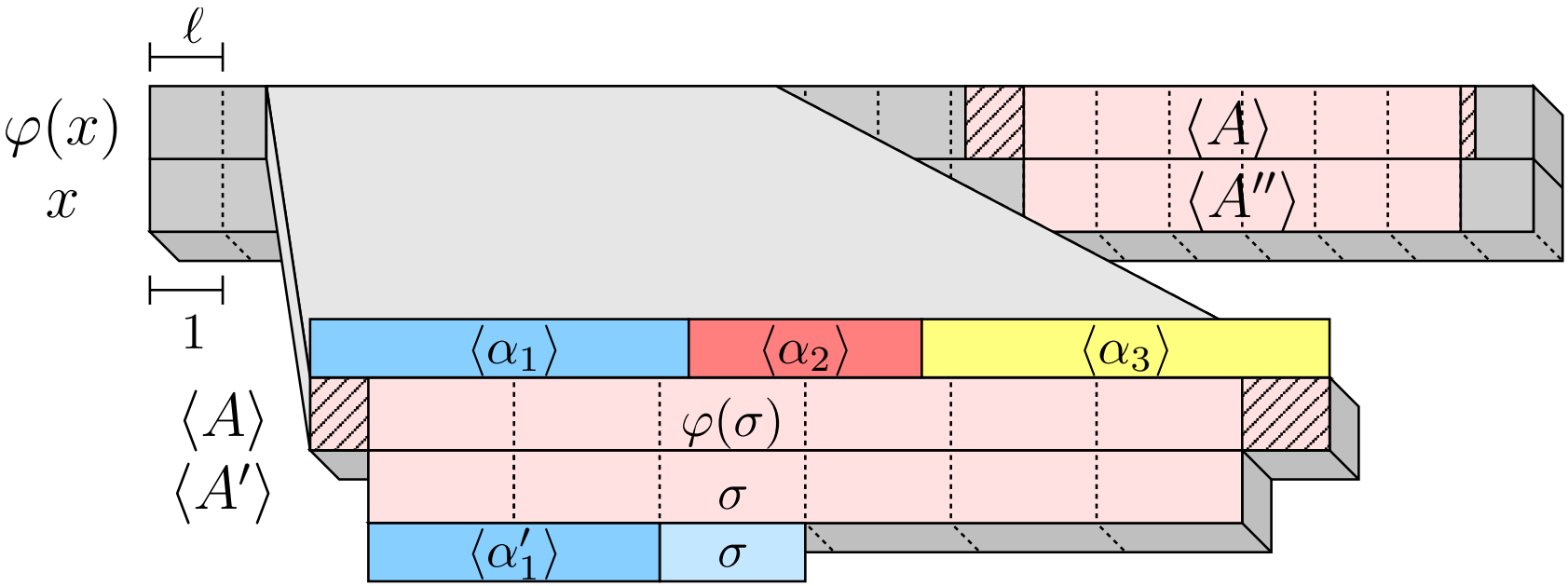

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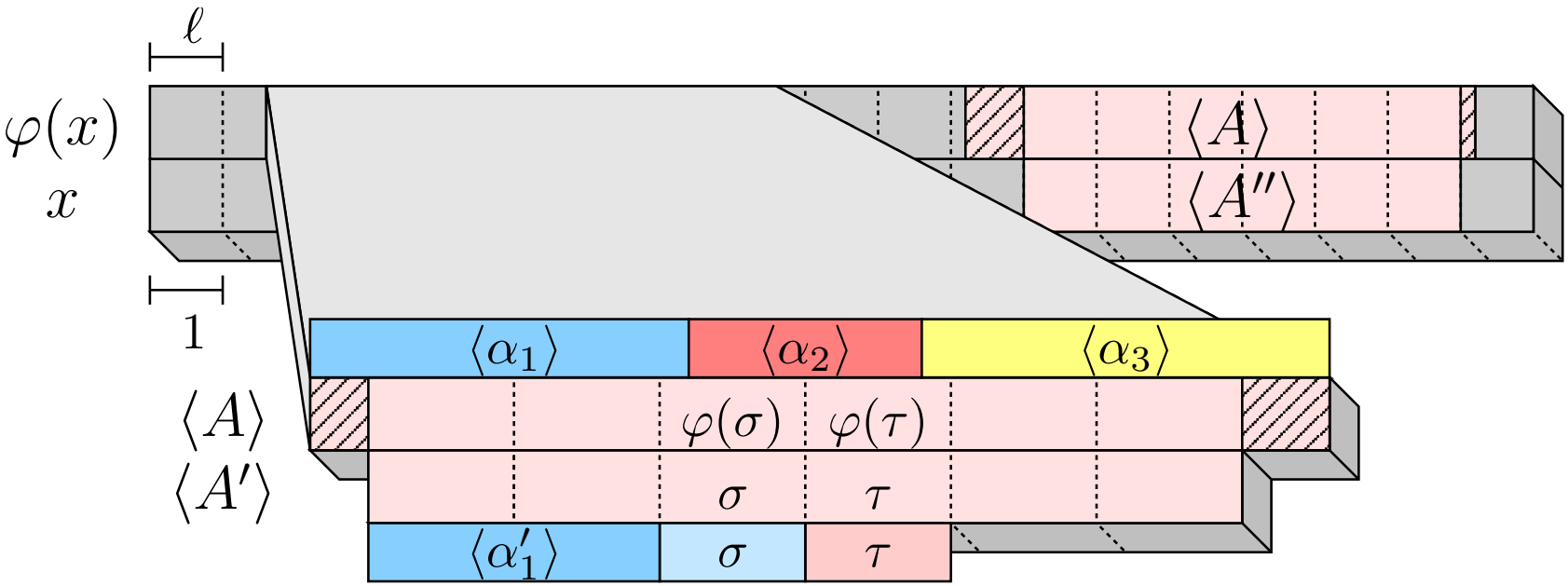

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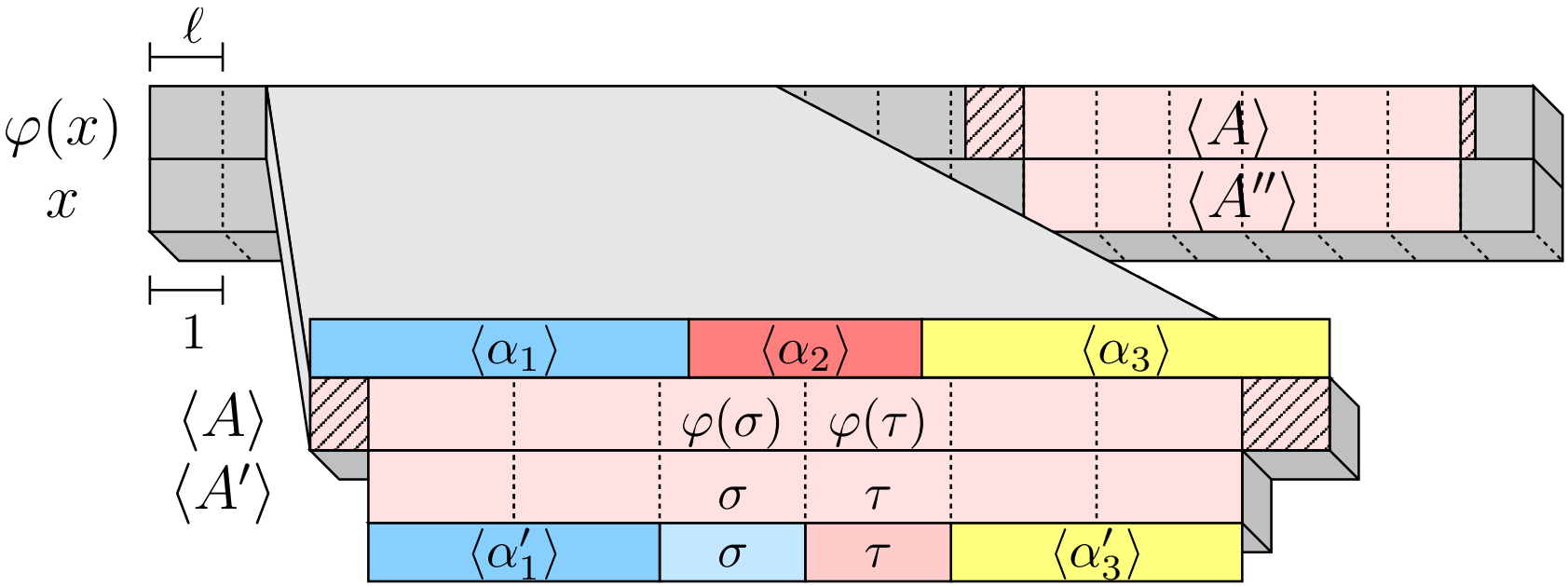

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Overlap-free Blockcodes

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- optimal compression of a string and its ℓ-blockcoding:

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  - edit grammars [CLL+02]