Treeningvõistlus

16.03.2018

1. Olgu funktsioon $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ ja tema osatuletised f_x ja f_y pidevad, kusjuures $f(0,0) = 0$ ning $|f_x(t,s)| \leq 2|t-s|$ ja $|f_y(t,s)| \leq 2|t-s|$ kõikide $t, s \in \mathbb{R}$ korral. Tõesta, et $|f(5,4)| \leq 1$.

2. Leia kõik sellised $n \in \mathbb{N}$, mille korral
$$\sum_{k=1}^{n} (k+2)^n = (n+3)^n.$$

3. Olgu $n, k \in \mathbb{N}$. Leia determinant
$$\begin{vmatrix} \binom{n}{0} & \binom{n}{1} & \cdots & \binom{n}{k} \\ \binom{n+1}{0} & \binom{n+1}{1} & \cdots & \binom{n+1}{k} \\ \cdots & \cdots & \cdots & \cdots \\ \binom{n+k}{0} & \binom{n+k}{1} & \cdots & \binom{n+k}{k} \end{vmatrix}.$$

4. Olgu $\gamma = e^{\frac{2\pi i}{9}} = \cos\frac{2\pi}{9} + i\sin\frac{2\pi}{9}$ (s.o. 9. astme algjuur). Leia neljanda astme täisarvuliste kordajatega polünoom, mille juureks on $\gamma + \gamma^{-1}$.

5. Olgu funktsioon $f : (a,b) \to \mathbb{R}$ *kumer*, s.t. $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$ kõikide $x, y \in (a,b)$ ja $\lambda \in [0,1]$ korral.

   (a) Tõesta, et iga $x \in (a,b)$ korral funktsioon $t \mapsto \dfrac{f(x+t)-f(x)}{t}$ on mittekahanev hulgas $(0, b-x)$.

   (b) Tõesta, et f on diferentseeruv kõikides intervalli (a,b) punktides, välja arvatud mingis ülimalt loenduvas hulgas.

Training competition

16.03.2018

1. Let a function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and its partial derivatives f_x and f_y be continuous. Assume that $f(0,0) = 0$ and $|f_x(t,s)| \le 2|t-s|$ and $|f_y(t,s)| \le 2|t-s|$ for all $t, s \in \mathbb{R}$. Prove that $|f(5,4)| \le 1$.

2. Find all $n \in \mathbb{N}$ such that
$$\sum_{k=1}^{n} (k+2)^n = (n+3)^n.$$

3. Let $n, k \in \mathbb{N}$. Find the determinant
$$\begin{vmatrix} \binom{n}{0} & \binom{n}{1} & \dots & \binom{n}{k} \\ \binom{n+1}{0} & \binom{n+1}{1} & \dots & \binom{n+1}{k} \\ \dots & \dots & \dots & \dots \\ \binom{n+k}{0} & \binom{n+k}{1} & \dots & \binom{n+k}{k} \end{vmatrix}.$$

4. Let $\gamma = e^{\frac{2\pi i}{9}} = \cos\frac{2\pi}{9} + i \sin\frac{2\pi}{9}$ (this is a primite ninth root of unity). Find a polynomial with integer coefficients of fourth degree having $\gamma + \gamma^{-1}$ as a root.

5. Let a function $f : (a,b) \to \mathbb{R}$ be *convex*, i.e. $f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y)$ for all $x, y \in (a,b)$ and $\lambda \in [0,1]$.

   (a) Prove that for all $x \in (a,b)$ the function $t \mapsto \dfrac{f(x+t) - f(x)}{t}$ is non-decreasing on $(0, b-x)$.

   (b) Prove that f is differentiable in all points of (a,b) except for some at most countable set of points.

Solutions.

1. Note that $f_x(x,y) = f_y(x,y) = 0$ whenever $x = y$. So differentiating the function $g(x) := f(x,x)$ we get
$$g'(x) = f_x(x,x)x' + f_y(x,x)x' = 0.$$
Hence, $f(4,4) = g(4) = 0$. Now
$$|f(5,4)| = \left|\int_4^5 f(x,4)dx + f(4,4)\right| \leq \int_4^5 |f(x,4)|dx \leq \int_4^5 2(x-4)dx = 1.$$

2. *Answer:* $n = 2,3$.

The cases $n = 2,3$ satisfy the equation. The cases $n = 1,4,5$ do not satisfy the equation because of different parity of sides. Let us show that for $n \geq 6$ one has
$$3^n + 4^n + \cdots + (n+2)^n < \left(\frac{1}{2^n} + \frac{1}{2^{n-1}} + \cdots + \frac{1}{2}\right)(n+3)^n < (n+3)^n.$$
So let us first show that
$$(n+2)^n \leq \frac{1}{2}(n+3)^n.$$
This is true for $n = 6$ because $2 \cdot 8^6 = 524288 < 531441 = 9^6$. Since
$$\frac{(n+3)^n}{(n+2)^n} = \left(1 + \frac{1}{n+2}\right)^n = \left(1 + \frac{1}{n+2}\right)^{n+2}\left(1 + \frac{1}{n+2}\right)^{-2}$$
is increasing, it is also true for all $n \geq 6$.

Let us prove now by induction that
$$(n+3-k)^n < \frac{1}{2^k}(n+3)^n$$
for $k \geq 2$. Assume
$$(n+3-(k-1))^n < \frac{1}{2^{k-1}}(n+3)^n.$$
Since
$$\frac{n+3-k}{n+3-(k-1)} \leq \frac{n+2}{n+3},$$
one has
$$\left(\frac{n+3-k}{n+3-(k-1)}\right)^n \leq \frac{(n+2)^n}{(n+3)^n} < \frac{1}{2}.$$
The assumption of the induction now yields the claim.

3. *Answer: 1.*

Denote the corresponding matrix by $A(n,k)$. Note that for $n = 0$ we get a lower triangular matrix, so
$$|A_{0,k}| = \begin{vmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & 1 & 0 & \dots & 0 \\ 1 & 2 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & n & \dots & \dots & 1 \end{vmatrix} = 1.$$

Note that

$$A(n+1,k) = A(n,k) \cdot \begin{pmatrix} 1 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix},$$

because $\binom{m}{i} + \binom{m}{i+1} = \binom{m+1}{i+1}$. So $|A(n+1,k)| = |A(n,k)| \cdot 1 = 1$.

4. Denote $\alpha_n = \gamma^n + \gamma^{-n}$ and $\alpha = \alpha_1$.
We have $\gamma^9 - 1 = 0$ and $\gamma - 1 \neq 0$, so

$$\gamma^8 + \gamma^7 + \cdots + \gamma + 1 = 0.$$

Divide it by γ^4 to get

$$\alpha_4 + \alpha_3 + \alpha_2 + \alpha + 1 = 0.$$

Note that $\alpha^2 = \alpha_2 + 2$, $\alpha^3 = \alpha_3 + 3\alpha$, and $\alpha^4 = \alpha_4 + 4\alpha_2 + 6$. From this it is easy to get

$$\alpha^4 + \alpha^3 - 3\alpha^2 - 2\alpha - 11 = 0.$$

5. (a) Let $0 < t < s$. Then

$$x + t = \lambda x + (1-\lambda)(x+s)$$

for $\lambda = 1 - \frac{t}{s}$. It remains to apply convexity.

(b) Note that for $s < u < t$ one has

$$\frac{f(u) - f(s)}{u - s} \le \frac{f(t) - f(u)}{t - u}.$$

Hence also for $s < u \le v < t$ one has

$$\frac{f(u) - f(s)}{u - s} \le \frac{f(t) - f(v)}{t - v}.$$

In other words, denoting

$$g(x,t) = \frac{f(x+t) - f(x)}{t}$$

we get that $g(x,s) \ge g(y,t)$ whenever $x > y$. This means that f has one-sided derivatives at every point and

$$f'_-(x) \le f'_+(x) \le f'_-(y) \le f'_+(y)$$

for all $y > x$. If f is not differentiable at x, then there is $q_x \in \mathbb{Q}$ such that

$$f'_-(x) < q_x < f'_+(x).$$

Note that q_x are distinct for distinct values of x.