Matemaatika võistlus

Tartu, 19.10.2018

1. Rahuldagu tõkestatud jada $(x_n)_{n=1}^{\infty} \subset \mathbb{R}$ tingimust $x_n - 2x_{n+1} + x_{n+2} \to 0$. Tõesta, et $x_n - x_{n+1} \to 0$.
2. Rahuldagu funktsioon $f : [0,1] \to \mathbb{R}$ võrratust
$$f\left(\frac{x+y}{2}\right) \le f(x) + f(y)$$
kõikide $x, y \in [0,1]$ korral ning olgu $f(0) = f(1) = 0$.
   (a) Tõesta, et $f(x) \ge 0$ iga $x \in [0,1]$ korral.
   (b) Tõesta, et $f(x) = 0$ lõpmata paljudes punktides $x \in [0,1]$.
   (c) Tuua näide ülesande tingimust rahuldavast mittekonstantsest funktsioonist f.
3. Olgu $p > 2$ algarv. Leia arvude
$$1 + \sum_{n=1}^{\frac{p-1}{2}} \binom{2n}{n} c^n, \quad c \in \mathbb{N},$$
kõik võimalikud jäägid jagamisel arvuga p.
4. Olgu $n \in \mathbb{N}$, olgu $P_1, \ldots, P_n$ nullist erinevad reaalarvuliste kordajatega polünoomid vastavalt astmetega $m_1, \ldots, m_n$ ning olgu $a_1, \ldots, a_n$ paarikaupa erinevad reaalarvud. Tõesta, et võrrandil
$$P_1(x)e^{a_1 x} + P_2(x)e^{a_2 x} + \cdots + P_n(x)e^{a_n x} = 0$$
on ülimalt $m_1 + \cdots + m_n + n - 1$ reaalarvulist lahendit.
5. Kehtigu reaalarvuliste $n \times n$ ruutmaatriksite A, B ja C korral $A^3 = -I$ ja $BA^2 + BA = C^6 + C + I$ (siin I on ühikmaatriks) ning olgu C sümmeetriline. Tõesta, et n ei saa olla paaritu.

Math competition

Tartu, 19.10.2018

1. A bounded sequence $(x_n)_{n=1}^{\infty} \subset \mathbb{R}$ is such that $x_n - 2x_{n+1} + x_{n+2} \to 0$. Prove that $x_n - x_{n+1} \to 0$.
2. The function $f : [0,1] \to \mathbb{R}$ is such that
$$f\left(\frac{x+y}{2}\right) \leq f(x) + f(y)$$
for all $x, y \in [0,1]$ and also $f(0) = f(1) = 0$.
    (a) Prove that $f(x) \geq 0$ for all $x \in [0,1]$.
    (b) Prove that $f(x) = 0$ for infinitely many $x \in [0,1]$.
    (c) Provide an example of a non-constant function f.
3. Let $p > 2$ be a prime number. Find all possible residues modulo p of numbers
$$1 + \sum_{n=1}^{\frac{p-1}{2}} \binom{2n}{n} c^n, \quad c \in \mathbb{N}.$$
4. Let $n \in \mathbb{N}$, let $P_1, \ldots, P_n$ be non-zero polynomials with real coefficients of degrees $m_1, \ldots, m_n$ respectively, and let $a_1, \ldots, a_n$ be pairwise distinct real numbers. Prove that the equation
$$P_1(x)e^{a_1 x} + P_2(x)e^{a_2 x} + \cdots + P_n(x)e^{a_n x} = 0$$
has at most $m_1 + \cdots + m_n + n - 1$ real solutions.
5. Real $n \times n$ matrices A, B, and C are such that $A^3 = -I$ and $BA^2 + BA = C^6 + C + I$ (here I denotes a unit matrix). Also, C is symmetric. Prove that n cannot be odd.

1. Denote $a_n := x_n - x_{n+1}$ and assume that $a_n \not\to 0$. We can find $\varepsilon > 0$ and a subsequence of (a_n) which lies outside of $(-\varepsilon, \varepsilon)$. In that, since (a_n) is bounded, we can find a further subsequence, which converges to some $a \neq 0$. In that, we can find a subsequence (a_{n_k}) such that x_{n_k} goes to some $x \in \mathbb{R}$. So $x_{n_k+1} = x_{n_k} - a_{n_k} \to x - a$. But since $a_n - a_{n+1} \to 0$, not only $a_{n_k} \to a$ but also $a_{n_k+m} \to a$ for any $m \in \mathbb{N}$. It follows that $x_{n_k+m} \to x - ma$ for any $m \in \mathbb{N}$. This contradicts the boundedness of (x_n).

2. (a) Assume $f(x) < 0$ for some $x \in [0,1]$. We have

$$f(\frac{x}{2}) \leq f(\frac{x}{4}) + f(\frac{3x}{4}) \leq f(0) + f(\frac{x}{2}) + f(\frac{x}{2}) + f(x) < 2f(\frac{x}{2}),$$

so $f(\frac{x}{2}) > 0$. But

$$f(\frac{x}{2}) \leq f(0) + f(x) < 0.$$

(b) It follows that if $f(x) = f(y) = 0$, then also $f(\frac{x+y}{2}) = 0$. Starting from $f(0) = f(1) = 0$, we arrive at $f(x) = 0$ for any $x = a_1 + a_2 2^{-1} + a_3 2^{-2} + \cdots + a_n 2^{-n}$ for some $n \in \mathbb{N}$ and $(a_i) \in \{0,1\}^n$.

(c) Denote the set of all x as above by A. Put $f(y) = 0$ if $y \in A$ and $f(y) = 1$ otherwise. Clearly, A is closed under operation $x, y \mapsto \frac{x+y}{2}$. So the inequality is satisfied for all $x, y \in A$. If one of x, y is not in A, then also $f(x) + f(y) \geq 1 + 0 = 1 \geq f(\frac{x+y}{2})$.

3. Note that the binomial expansion

$$(1+a)^{\frac{p-1}{2}} = 1 + \sum_{n=1}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{n} a^n$$

looks very similar to the expression in question. So let us compare $\binom{2n}{n}$ to $\binom{\frac{p-1}{2}}{n}$ modulo p. But first note that $(-2)(\frac{p-1}{2} - i) = 2i - (p-1) \equiv 2i + 1$. We have

$$\binom{2n}{n} = \frac{(2n)!}{(n!)^2} = \frac{2^n(1 \cdot 3 \cdot \cdots \cdot (2n-1))}{n!} \equiv \frac{(-4)^n \frac{p-1}{2}(\frac{p-1}{2} - 1) \cdot \cdots \cdot (\frac{p-1}{2} - (n-1))}{n!} = (-4)^n \binom{\frac{p-1}{2}}{n}$$

(the congruence is legal, because clearly $n!$ is invertible modulo p). Thus

$$1 + \sum_{n=1}^{\frac{p-1}{2}} \binom{2n}{n} c^n \equiv 1 + \sum_{n=1}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{n} (-4)^n c^n = (1-4c)^{\frac{p-1}{2}} \equiv (4(4^{-1} - c))^{\frac{p-1}{2}} \equiv (4^{-1} - c)^{\frac{p-1}{2}},$$

because $4^{\frac{p-1}{2}} = 2^{p-1} \equiv 1$. So the question reduces to finding the possible values of $b^{\frac{p-1}{2}}$ modulo p when b runs in $\mathbb{N}$. By Euler's criterion these are -1, 0, and 1.

4. We prove by induction on n. Base $n = 1$ is clear. Denote

$$g(x) := P_1(x)e^{a_1 x} + P_2(x)e^{a_2 x} + \cdots + P_n(x)e^{a_n x}.$$

Assume that $g(x)$ has N real zeroes. Then $e^{-a_n x} g(x)$ has as many and $(e^{-a_n x} g(x))'$ has at least $N - 1$. Continuing in this fashion, we get that $(e^{-a_n x} g(x))^{(m_n+1)}$ has at least $N - m_n - 1$ real zeroes. But $(e^{-a_n x} g(x))^{(m_n+1)} = (e^{-a_n x} g(x) - P_n(x))^{(m_n+1)}$. The latter function satisfies the induction hypothesis for $n - 1$ (the degrees of the new polynomials do not exceed those of $P_1, \ldots, P_{n-1}$). So

$$N - m_n - 1 \leq m_1 + \cdots + m_{n-1} + n - 2,$$

from which the claim follows.

5. Since C is symmetric, it is diagonalizable. So it cannot be singular, because the polynomial $x^6 + x + 1$ does not have real roots. So $BA(A+I)$ is regular and so is $A+I$. But $(A^2 - A+I)(A+I) = A^3+I = 0$ and hence $A^2 - A+I = 0$. The polynomial $x^2 - x+1$ does not have real roots. So A does not have real eigenvalues. But any matrix of odd order must have at least one real eigenvalue (by the intermediate value theorem applied to the characteristic polynomial).

Kommentaarid ja osalised hindamisskeemid:

1. Paljudel oli arutelu, et $a_n - a_{n+1} \to 0$ annab, et (a_n) on Cauchy (ja seega koondub), tihti järgmise "tõestusega": $|a_n - a_{n+p}| \leq |a_n - a_{n+1}| + |a_{n+1} - a_{n+2}| + \cdots + |a_{n+p-1} - a_{n+p}| \to 0$.

   See on vale. Lihtsaks kontranäideks on harmoonilise rea osasummad $S_n = 1 + \cdots + \frac{1}{n} \to \infty$. Veelgi enam, selle näite põhjal saab natuke mõeldes konstrueerida ka tõkestatud mittekoonduva jada sama omadusega.

   Skeem:

   7p: Tehtud eeldusel või valesti tõestades, et (a_n) koondub.

   6p: Sama, aga pole aru saadud, et see eeldus on tehtud.

   1p: Tehtud eeldusel, et (x_n) koondub.

2. Osa a) palju lihtsam tõestus on muidugi võtta $x = y$.

   Skeem: 20p = a) 7p + b) 7p + c) 6p.

3. Skeem: 3p õige vastuse eest. Triinule lisaks 2p avaldiseni $(-1)^{\frac{p-1}{2}}$ jõudmise eest, kust oleks võinud tekkida idee vaadata lahenduses toodud binoomvalemit.

4. Skeem: 3p punkti tuletise $(e^{ax}P(x))'$ vaatamise eest, mis on oluline samm õiges suunas.

5. Skeem: 2p maatriksi C omaväärtuste vaatamise eest.