Tartu Ülikooli üliõpilaste matemaatikaolümpiaad

25.05.2012

1. Kehtigu funktsiooni $f : [1, \infty) \to \mathbb{R}$ korral tingimused $f(1) = 1$ ja
$$f'(x) = \frac{1}{x^2 + f(x)^2} \quad \forall x \in [1, \infty).$$
Tõestage, et $a := \lim_{x\to\infty} f(x)$ eksisteerib, kusjuures $a < 1 + \frac{\pi}{4}$.

2. Ütleme, et funktsioon $p : \mathbb{R}^2 \to [0, \infty)$ on *võrekujutus*, kui reaalarvude $x, y,\ x_1$ ja y_1 korral võrratustest $|x_1| \le |x_2|$ ja $|y_1| \le |y_2|$ järeldub $p(x_1, y_1) \le p(x_2, y_2)$.
Tähistagu $\|(x, y)\| = \sqrt{x^2 + y^2}$ eukleidilist normi tasandil $\mathbb{R}^2$. Fikseerime sirge l tasandil $\mathbb{R}^2$ ja arvu $\varepsilon \ne 0$ ning defineerime
$$p_{l,\varepsilon}(x, y) = \|(x, y)\| + \varepsilon \operatorname{dist}(x, y, l),$$
kus $\operatorname{dist}(x, y, l) := \min_{(\xi,\eta)\in l} \|(x - \xi, y - \eta)\|$ on punkti (x, y) kaugus sirgeni l. Milliste paaride (l, ε) korral on $p_{l,\varepsilon}$ võrekujutus?

3. Tõestage, et hulk
$$\left\{ x > 0 : \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n + x} \in \mathbb{Q} \right\}$$
on lõpmatu.

4. (a) Kas leidub selline lõplik reaalarvude hulk A, milles on vähemalt 4 elementi, et tema suvaliste paarikaupa erinevate elementide $a,\ b,\ c$ ja d korral ka $ab + cd \in A$.
   (b) Kas leidub positiivsete reaalarvude hulk A, mis rahuldab samu tingimusi?

5. Olgu
$$f(x) = \frac{1 - x + x^2}{1 + x + x^2}.$$
Leidke $f^{(n)}(0)$ iga $n \in \mathbb{N}$ korral.

6. Kas leidub 12×12 ruutmaatriks A, mis koosneb ainult arvudest -1, 0 ja 1 ning mille determinant $\det A$ on 2012?

Tartu University student olympiad in mathematics

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1. Let a function $f : [1, \infty) \to \mathbb{R}$ be such that $f(1) = 1$ and
$$f'(x) = \frac{1}{x^2 + f(x)^2} \quad \forall x \in (1, \infty).$$
Prove that $a := \lim_{x\to\infty} f(x)$ exists and $a < 1 + \frac{\pi}{4}$.

2. Let us say that a function $p : \mathbb{R}^2 \to [0, \infty)$ is a *lattice map*, if for all real numbers $x, y,\ x_1$, and y_1 the inequalities $|x_1| \le |x_2|$ and $|y_1| \le |y_2|$ imply $p(x_1, y_1) \le p(x_2, y_2)$.
Let $\|(x, y)\| = \sqrt{x^2 + y^2}$ denote the Euklidian norm on $\mathbb{R}^2$. Fix a straight line l in $\mathbb{R}^2$ and $\varepsilon \ne 0$. Then define
$$p_{l,\varepsilon}(x, y) = \|(x, y)\| + \varepsilon \operatorname{dist}(x, y, l),$$
where $\operatorname{dist}(x, y, l) := \min_{(\xi,\eta)\in l} \|(x - \xi, y - \eta)\|$ is a distance from the point (x, y) to the line l. For which pairs (l, ε) does $p_{l,\varepsilon}$ happen to be a lattice map?

3. Prove that the set
$$\left\{ x > 0 : \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n + x} \in \mathbb{Q} \right\}$$
is infinite.

4. (a) Does there exist a finite set $A \subset \mathbb{R}$ having at least 4 elements such that for all distinct $a, b, c, d \in A$ one also has $ab + cd \in A$?
   (b) Is there a set $A \subset (0, \infty)$ satisfying the conditions above?

5. Let
$$f(x) = \frac{1 - x + x^2}{1 + x + x^2}.$$
Find $f^{(n)}(0)$ for all $n \in \mathbb{N}$.

6. Is there a 12×12 matrix A consisting of numbers -1, 0, and 1 such that $\det A = 2012$?

Solutions

1. For $x > 1$ we have $f'(x) > 0$ and

$$f(x) = f(1) + \int_1^x f'(t)dt > f(1) = 1.$$

Hence,

$$\lim_{x\to\infty} f(x) = 1 + \int_1^x \frac{dt}{t^2 + f(t)^2} < 1 + \int_1^x \frac{dt}{t^2+1} = 1 + \arctan(t)\Big|_1^x \underset{x\to\infty}{\longrightarrow} 1 + \frac{\pi}{2} - \frac{\pi}{4} = 1 + \frac{\pi}{4}.$$

Since f is increasing, $\lim_{x\to\infty} f(x)$ exists and is not greater than $1 + \frac{\pi}{4}$. Now the strict inequality follows from

$$\lim_{x\to\infty} f(x) = 1 + \int_1^\infty \frac{dt}{t^2 + f(t)^2} < 1 + \int_1^\infty \frac{dt}{t^2+1} = 1 + \frac{\pi}{4}.$$

2. Only pairs (l, ε) where $\varepsilon > 0$ and l is either x- or y-axis yield the lattice map.

First, notice that for $\varepsilon > 0$ both $p_{x,\varepsilon} = \|(x,y)\| + \varepsilon|y|$ and $p_{y,\varepsilon} = \|(x,y)\| + \varepsilon|x|$ certainly are lattice maps.

Observe that a lattice map p must be symmetric with respect to both axes, i.e., $|x_1| = |x_2|$ and $|y_1| = |y_2|$ imply $p(x_1, y_1) = p(x_2, y_2)$. Therefore, l must be symmetric with respect to both axes, i.e., it must be one of them. Indeed, let $(x, y) \in l$ then $p(x, y) = \|(x, y)\|$ and, e.g.,

$$p(x, y) = p(-x, y) = \|(-x, y)\| + \varepsilon \operatorname{dist}(-x, y, l) = \|(x, y)\| + \varepsilon \operatorname{dist}(-x, y, l),$$

so that $\operatorname{dist}(-x, y, l) = 0$, or equivalently $(-x, y) \in l$.

Assume now that $\varepsilon < 0$ and, e.g., $p(x, y) = \|(x, y)\| - |\varepsilon||y|$. To show that p is not a lattice map we are going to find $x > 0$ such that $p(x, 1) < p(x, 0)$. The latter means $\sqrt{x^2+1} - |\varepsilon| < x$, i.e., $\sqrt{x^2+1} - x < |\varepsilon|$. Fix $\delta < \min\{1, |\varepsilon|\}$. Then $\sqrt{x^2+1} - x = \delta$ if and only if $x = (1-\delta^2)/(2\delta)$. This x suits us.

3. Fix $x \geq 0$. Then $1/(n + x)$ converges to 0 monotonically. Hence, the alternating series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+x}$ converges. Moreover,

$$\left| \sum_{n=m}^{\infty} \frac{(-1)^{n+1}}{n+x} \right| \leq \frac{1}{m+x} \leq \frac{1}{m},$$

so that it converges uniformly on $[0, \infty)$. Therefore, the function

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+x}$$

is continuous on $[0, \infty)$. Now clearly $f(0) > 1/2$ (actually $f(0) = \ln 2$ but it is not that important) and $f(1) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1} = -\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - f(0)$. The intermediate value theorem now gives $(f(1), f(0)) \subset f[(0, 1)]$, which is enough.

4. (a) Yes. Take $A = \{0, 1/2, 1, 2\}$.

   (b) No. Assume that there is such A.

   i. A contains at most 1 element greater than 1.
   Indeed, if $a > b > 1$ were the greatest elements in A then $ab + cd > ab > a$, so that $ab + cd \notin A$.

ii. If A consists of 4 elements then $\max A \le 1$.
Indeed, assume $A = \{a, b, c, d\}$ and $a > 1 \ge b > c > d$. Since

$$(ab + cd) - (ac + bd) = (a - d)(b - c) > 0,$$
$$(ac + bd) - (ad + bc) = (a - b)(c - d) > 0,$$

we have

$$ab + cd > ac + bd > ad + bc > ad > d,$$

so that $ab + cd = a$, $ac + bd = b$, and $ad + bc = c$. But then

$$a - c = ab + cd - (ad + bc) = (a - c)(b - d) < a - c$$

because $b - d < b \le 1$.

iii. Hence, A contains at least 4 elements not greater than 1.

Now take two elements $a > b$ in A with the minimal difference $a - b$. We know that A contains some other two elements $c > d$ such that $c, d \le 1$. Then $c - d < 1$. Now

$$(ac + bd) - (ad + bc) = (a - b)(c - d) < a - b,$$

which contradicts the minimality of $a - b$.

5. Let $|x| < 1$. Then

$$f(x) = 1 - \frac{2x}{1 + x + x^2} = 1 - \frac{2x(1 - x)}{1 - x^3} = 1 + (2x^2 - 2x)(1 + x^3 + x^6 + \dots) =$$
$$= f(0) - 2x + 2x^2 - 2x^4 + 2x^5 - 2x^7 + 2x^8 - \dots.$$

Hence,

$$f^{(n)}(0) = \begin{cases} -2n! & \text{if } n = 3k - 2, \\ 2n! & \text{if } n = 3k - 1, \\ 0 & \text{if } n = 3k. \end{cases}$$

6. Yes. Take

$$A = \begin{pmatrix} 1 & -1 & & & & \\ 1 & 1 & -1 & & 0 & \\ 1 & 1 & 1 & -1 & & \\ \dots & \dots & \dots & \dots & \dots & \\ 1 & 1 & 1 & 1 & \dots & -1 \\ 0 & a_0 & a_1 & a_2 & \dots & a_{10} \end{pmatrix}$$

where $\overline{a_{10}a_9 \dots a_0}$ is the binary representation of 2012, i.e., $a_5 = a_1 = a_0 = 0$ and $a_i = 1$ for other i-s.

To prove this, let us solve the equation $A(x_1, x_2, \dots, x_{12})^T = b^T$ with $b = (0, 0, \dots, 0, 1)$ in two ways. First, we have $x_2 = x_1$, $x_3 = 2x_1$, $x_4 = 4x_1$, $\dots$, $x_{12} = 2^{10}x_1$, and the last row gives $2012x_1 = 1$ or $x_1 = 1/2012$.

On the other hand, the Cramer's rule implies

$$x_1 = (\det A)^{-1} \cdot \begin{vmatrix} 0 & -1 & & & & \\ 0 & 1 & -1 & & 0 & \\ 0 & 1 & 1 & -1 & & \\ \dots & \dots & \dots & \dots & \dots & \\ 0 & 1 & 1 & 1 & \dots & -1 \\ 1 & a_0 & a_1 & a_2 & \dots & a_{10} \end{vmatrix} = -(\det A)^{-1} \cdot \begin{vmatrix} -1 & & & & \\ 1 & -1 & & 0 & \\ 1 & 1 & -1 & & \\ \dots & \dots & \dots & \dots & \\ 1 & 1 & 1 & \dots & -1 \\ a_0 & a_1 & a_2 & \dots & a_{10} \end{vmatrix} = (\det A)^{-1}.$$

Hence, $\det A = 2012$.