Tartu Ülikooli üliõpilaste matemaatikaolümpiaad

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1) *Poolvõre* X on osaliselt järjestatud hulk, milles igal kahel elemendil $x, y \in X$ leidub vähim ülemine tõke $x \vee y \in X$. Osahulk $Y \subset X$ on *alampoolvõre*, kui $x, y \in Y$ korral $x \vee y \in Y$. On teada, et poolvõre X kõikide alampoolvõrede hulk $L(X)$ on samuti poolvõre hulgateoreetilise järjestuse suhtes (s.t. $Y_1 \leq Y_2 \Leftrightarrow Y_1 \subset Y_2$). Poolvõred X ja Y on *isomorfsed*, kui leidub bijektiivne kujutus $f : X \to Y$ nii, et $f(x_1) \vee f(x_2) = f(x_1 \vee x_2)$ kõikide $x_1, x_2 \in X$ korral.

   Leia sellised poolvõred X ja Y nii, et $L(X)$ ja $L(Y)$ on isomorfsed, kuid X ja Y ei ole.

2) Olgu p polünoom astmega $n \geq 1$, kusjuures leidub $a \in \mathbb{R}$ nii, et

$$p(a) + \frac{p''(a)}{2!} + \frac{p^{(4)}(a)}{4!} + \cdots + \frac{p^{(2n)}(a)}{(2n)!} = 0.$$

   Tõesta, et leidub $b \in \mathbb{R}$ nii, et $p(b) = 0$.

3) Olgu antud 3 mittekomplanaarset vektorit $a, b, c \in \mathbb{R}^3$. Kas alati leiduvad vektorid $x, y, z \in \mathbb{R}^3$ nii, et $x \times y = c$, $x \times z = b$ ja $y \times z = a$?

4) Arvuta

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)3^n}.$$

5) Ruutmaatriksite A ja B korral $AB = BA$, $A^{2010} = E$ ja $B^{2011} = E$, kus E on ühikmaatriks. Tõesta, et maatriks $A + B + E$ on regulaarne.

6) Olgu $f : \mathbb{R} \to \mathbb{R}$ pidev funktsioon, kusjuures

$$f(a) + f(b) \geq \int_a^b f(x)^2 dx$$

   kõikide $a, b \in \mathbb{R}$ korral. Tõesta, et $f(x) = 0$ iga $x \in \mathbb{R}$ korral.

Solutions

1) Consider two semilattice orders $\leq_1$ and $\leq_2$ on the set $\{1,2,3,4\}$ such that $2 \leq_1 1, 4 \leq_1 3$ with 1 and 4 being $\leq_1$-incomparable, and $1, 4 \leq_2 3 \leq_2 2$ with 1 and 4 being $\leq_2$-incomparable. In both cases the set of all subsemilattices is just the power set $2^{\{1,2,3,4\}}$ minus the sets $\{1,4\}$ and $\{1,2,4\}$. Hence, the respective semilattices of subsemilattices coincide. Since the semilattice isomoprphism respects the order, it is clear that the original semilattices are not isomorphic (e.g. both 1 and 4 would have to be mapped to 3).

2) The Taylor expansions for $p(x+1)$ and $p(x-1)$ give us

$$p(x+1) = p(x) + p'(x) + \frac{p''(x)}{2!} + \cdots + \frac{p^{(2n)}(a)}{(2n)!}$$

and

$$p(x-1) = p(x) - p'(x) + \frac{p''(x)}{2!} - \cdots + \frac{p^{(2n)}(a)}{(2n)!},$$

so that

$$\frac{p(x-1)+p(x+1)}{2} = p(x) + \frac{p''(x)}{2!} + \cdots + \frac{p^{(2n)}(a)}{(2n)!}.$$

It is now clear that if $p(x)$ is always positive (negative) then so is

$$\frac{p(x-1)+p(x+1)}{2}.$$

3) No. This follows from the fact that

$$(y \times z, z \times x, x \times y) = (x, y, z)^2.$$

(Here $(x, y, z) = (x, y \times z)$ denotes the triple product.) Hence, $(y \times z, x \times z, x \times y) \leq 0$ and the equality would mean $(a, b, c) < 0$ which is not always the case.

To prove the fact, observe that $(x, y, z) = |A|$, where

$$A = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix}$$

and $(y \times z, z \times x, x \times y) = |B|$, where

$$B = \begin{pmatrix} \begin{vmatrix} y_2 & y_3 \\ z_2 & z_3 \end{vmatrix} & - \begin{vmatrix} y_1 & y_3 \\ z_1 & z_3 \end{vmatrix} & \begin{vmatrix} y_1 & y_2 \\ z_1 & z_2 \end{vmatrix} \\ - \begin{vmatrix} x_2 & x_3 \\ z_2 & z_3 \end{vmatrix} & \begin{vmatrix} x_1 & x_3 \\ z_1 & z_3 \end{vmatrix} & - \begin{vmatrix} x_1 & x_2 \\ z_1 & z_2 \end{vmatrix} \\ \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} & - \begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix} & \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} \end{pmatrix}.$$

But the elements of B are exactly the cofactors to the elements of A. Hence, we can divide the elements of B by $|A|$ and obtain the matrix A^{-1}. Therefore $|A^{-1}| = |B| \,/\, |A|^3$, so that $|B| = |A|^2$, as needed.

4) Integrate the power series

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}, \quad |z| < 1,$$

twice. We obtain

$$\sum_{n=0}^{\infty} \frac{z^{n+1}}{n+1} = \int_0^z \frac{du}{1-u} = -\ln(1-z)$$

and

$$\sum_{n=0}^{\infty} \frac{z^{n+2}}{(n+1)(n+2)} = \int_0^z (-\ln(1-u))du = z + (1-z)\ln(1-z),$$

so that

$$\sum_{n=1}^{\infty} \frac{z^n}{n(n+1)} = \frac{1}{z}\sum_{n=0}^{\infty} \frac{z^{n+2}}{(n+1)(n+2)} = 1 + \frac{1-z}{z}\ln(1-z).$$

Putting $z = 1/3$ we get

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)3^n} = 1 + 2\ln 2 - 2\ln 3.$$

5) Assume that $A + B + E$ is singular, so that there is a vector $x \neq 0$ such that $(A+E)x = -Bx$. Then

$$-B^2x = B(A+E)x = (A+E)Bx = -(A+E)^2x.$$

By induction, we get

$$(A+E)^k x = (-1)^k B^k x$$

for all integer $k \geq 2$.

Since $B^{2011} = E$, we have

$$((A+E)^{2011} + E)x = 0. \tag{1}$$

On the other hand we have

$$(A^{2010} - E)x = 0. \tag{2}$$

Let us show that polynomials $(z+1)^{2011}+1$ and $z^{2010}-1$ are mutually prime. Assume that $z_0 \in \mathbb{C}$ is their common root. Then $|z_0| = 1$ and $|z_0 + 1| = 1$. Therefore on the complex plane, the complex numbers 0, 1, and z_0+1 pose as the vertices of an equilateral triangle. Hence $\arg(z_0 + 1)$ is either $\pi/3$ or $-\pi/3$. Since $2011 \equiv 1$ modulo 3, the same is true for $\arg(z_0 + 1)^{2011}$, which contradicts $(z_0 + 1)^{2011} = -1$.

Hence, we can find the polynomials p and q such that

$$p(z)[(z+1)^{2011} + 1] + q(z)(z^{2010} - 1) = 1$$

for all z. Therefore,

$$p(A)[(A+E)^{2011} + E] + q(A)(A^{2010} - E) = E.$$

But (1) and (2) imply that

$$\big(p(A)[(A+E)^{2011} + E] + q(A)(A^{2010} - E)\big)\, x = 0,$$

so that $x = Ex = 0$. Therefore, $A + B + E$ is regular.

6) Consider two cases.

First case: for every $a \in \mathbb{R}$ one has

$$\int_a^\infty f^2(x)dx \le f(a).$$

Denote $F(x) := \int_x^\infty f^2(t)dt$, then

$$F'(x) = -f^2(x) \le -\left(\int_x^\infty f^2(t)dt\right)^2 = -F^2(x).$$

We show $F \equiv 0$, then it would follow that $f \equiv 0$. Assume that there exists $x_0 \in \mathbb{R}$ such that $F(x_0) > 0$. Then also $F(x) > 0$ for all $x < x_0$. Integrating the inequality

$$\frac{F'(t)}{F^2(t)} \le -1$$

over the interval $[x, x_0]$, we get

$$-\frac{1}{F(t)}\bigg|_x^{x_0} \le -(x_0 - x),$$

so that

$$f(x) \ge F(x) \ge \frac{1}{\frac{1}{F(x_0)} - (x_0 - x)}.$$

But then $f(x) \to \infty$ whenever x tends to $x_0 - 1/F(x_0)$ from above, which contradicts the continuity of f.

Second case: there exists $a \in \mathbb{R}$ such that

$$\int_a^\infty f^2(x)dx > f(a).$$

Then we can find $B > a$ such that for all $x \geq B$ one has

$$\int_a^x f^2(t)dt > f(a).$$

Using the assumption $f(x) + f(a) \geq \int_a^x f^2(t)dt$ we get

$$f(x) \geq \int_a^x f^2(t)dt - f(a) > 0.$$

Put $F(x) := \int_a^x f^2(t)dt - f(a)$. Then $F'(x) = f^2(x) \geq F^2(x)$. Similarly to the above we get

$$-\frac{1}{F(t)}\Big|_B^x \geq x - B,$$

so that

$$f(x) \geq F(x) \geq \frac{1}{\frac{1}{F(B)} + B - x}.$$

This means $f(x) \to \infty$ whenever x tends to $B + 1/F(B)$ from below, a contradiction.