Summability, Matrix Transformations and Their Applications

Doctoral School, Tartu University, Tartu, Estonia

13 October, 2011

Eberhard Malkowsky
Department of Mathematics
Faculty of Arts and Sciences
Fatih University
34500 Büyükçekmece, Istanbul
Turkey

cite{email: ema@Bankerinter.net}
Eberhard.Malkowsky@math.uni-giessen.de

1Research supported by the research project #174025 of the Serbian Ministry of Science, Technology and Environment
Mathematics Subject Classification: 46A45, 40C05
Key words and phrases: Sequence spaces, dual spaces, matrix transformations, Hausdorff measure of noncompactness, compact operators, Fredholm operators
1 Introduction

2 Matrix Transformations
   2.1 Notations
   2.2 $FK$, $BK$ and $AK$ spaces
   2.3 The dual spaces
   2.4 General results

3 Compact Operators
   3.1 The Hausdorff measure of noncompactness

4 Fredholm Operators on $c_0$

5 The Spaces of Strongly Summable and Bounded Sequences
6 Graphical Representations of Neighbourhoods

7 Wulff’s Crystals and Potential Surfaces

References
1 Introduction

The theory of matrix transformations is a wide field in summability; it deals with the characterisations of classes of matrix mappings between sequence spaces by giving necessary and sufficient conditions on the entries of the infinite matrices.

The most important applications are

• inclusion, Mercerian and Tauberian theorems
• characterisation of compact linear operators
• study of Fredholm operators given by matrices
• determination of Banach algebras of matrix transformations
• spectral theory, invertibility of operators, solvability of infinite systems of linear equations
Summability

The classical summability theory deals with a generalization of convergence of sequences and series. One original idea was to assign a limit to divergent sequences or series.

Classical methods of summability were applied to problems in analysis such as the analytic continuation of power series and improvement of the rate of convergence of numerical series.

This was achieved by considering a transform, given by an infinite matrix, rather than the original sequence or series; most popular are

- **Hausdorff** matrices, and their special cases
  - **Cesàro** matrices $C_\alpha$
  - **Euler** matrices $E_q$
  - **Hölder** matrices $H_\alpha$

- **Nörlund** matrices $(N, p)$
Concepts of summability

Let \( A = (a_{nk})_{n,k=1}^\infty \) be an infinite matrix, \( x = (x_k)_{k=1}^\infty \) be a sequence, \( e = (1, 1, 1, \cdots) \), \( A_n x = \sum_{k=1}^\infty a_{nk} x_k \) and \( Ax = (A_n x)_{n=1}^\infty \) be the sequence of the \( A \) transforms of \( x \). There are three concepts of summability.

- **Ordinary summability**: \( x \) is summable \( A \) if
  \[
  \lim_{n \to \infty} A_n x = \ell \text{ for some } \ell \in \mathbb{C}
  \]

- **Strong summability**: \( x \) is strongly summable \( A \) with index \( p > 0 \) if
  \[
  \lim_{n \to \infty} A_n (|x - \ell \cdot e|^p) = \lim_{n \to \infty} \sum_{k=1}^\infty a_{nk} |x_k - \ell|^p = 0 \text{ for some } \ell \in \mathbb{C}
  \]

- **Absolute summability**: \( x \) is absolutely summable \( A \) with index \( p > 0 \) if
  \[
  \sum_{n=1}^\infty |A_n x - A_{n-1} x|^p < \infty.
  \]
An example

Example 1.1 Let the matrix $A$ be given by $a_{nk} = 1/n$ for $1 \leq k \leq n$ and $a_{nk} = 0$ for $k > n$ ($n = 1, 2, \ldots$). Then the $A$ transforms of the sequence $x$ are the arithmetic means of the terms of $x$, that is,

$$\sigma_n = \frac{1}{n} \sum_{k=1}^{n} x_k$$

and $A$ defines the Cesàro method $C_1$ of order 1.

- Every convergent sequence is summable $C_1$ and the limit is preserved
- the divergent sequence $((-1)^k)_{k=1}^{\infty}$ is summable $C_1$ to 0
- strong summability of index 1 implies ordinary summability to the same limit; the converse is not true, in general
- absolute summability with index 1 implies ordinary summability.
2 Matrix Transformations

Let $X$ and $Y$ be subsets of the set $\omega$ of all complex sequences. The theory of matrix transformations deals with the characterisation of the class $(X, Y)$ of all infinite matrices that map $X$ into $Y$. So $A \in (X, Y)$ if and only if

\[(2.1) \quad A_{n}x \text{ converges for all } n \text{ and all } x \in X\]

and

\[(2.2) \quad Ax \in Y \text{ for all } x \in X.\]

We are going to study

- $FK$, $BK$ and $AK$ spaces
- dual spaces
- special classes of matrix transformations
2.1 Notations

\[ \ell_\infty \]
\[ c \]
\[ c_0 \]
\[ \phi \]
\[ \{ \text{the sets of all } \}
\[ \{ \text{bounded convergent null finite} \} \]
\[ \{ \text{sequences } x = (x_k)_k \} \]

\[ e_k = 1, \ (k = 1, 2, \ldots) \]

\[ e^{(n)} \ (n \in \mathbb{N}) \]

\[ e_k^{(n)} = \begin{cases} 
1 & (k = n) \\
0 & (k \neq n) 
\end{cases} \]
$B_X(r, x_0)$, $S_X(r, x_0)$ and $X'$

Let $(X, d)$ be a metric space. Then

- $B_X(r, x_0) = \{x \in X : d(x, x_0) < r\}$ is the open ball and $S_X(r, x_0) = \{x \in X : d(x, x_0) = r\}$ is the sphere in $X$ of radius $r > 0$ and centre in $x_0 \in X$; we write $B_X = B_X(1, 0)$ and $S_X = S_X(1, 0)$
- if $X$ is a linear metric space then $X'$ is the *continuous dual of $X$*, that is, the set of all continuous linear functionals on $X$.

$B(X, Y)$ and $X^*$

Let $X$ and $Y$ be normed spaces. Then

- $B(X, Y)$ is the space of all bounded linear operators $L : X \to Y$ with
  \[ \|L\| = \sup_{x \in S_X} \|L(x)\| \]
- $X^* = B(X, \mathbb{C})$ is the continuous dual of $X$ with
  \[ \|f\| = \sup_{x \in S_X} |f(x)| \quad \text{for all } f \in X^* \]
2.2 \( FK, BK \) and \( AK \) spaces

The theory of \( FK, BK \) and \( AK \) spaces from modern functional analysis is the most powerful tool in the characterisation of matrix transformations. The reason is that matrix transformations between \( FK \) spaces are continuous.

The theory, however, fails in some cases, for instance, when the initial sequence space has no Schauder basis, as in the determination of the class \((\ell_\infty, c)\) of coercive matrices.

In such cases, the method of the gliding hump from classical analysis is applied.
**FK and BK spaces ([27, p. 55])**

A subspace $X$ of $\omega$ is said to be an **FK space** if it is a Fréchet space, that is, a complete, locally convex, linear metric space, with continuous coordinates

$$P_n : X \rightarrow \mathbb{C} \ (n = 1, 2, \ldots) \text{ where } P_n(x) = x_n;$$

a **BK space** is an **FK** space with its metric given by a norm.

---

**AK property [27, Definition 4.2.13]**

An **FK** space $X \ni \phi$ is said to have **AK** if every sequence $x = (x_k)_{k=1}^{\infty} \in X$ has a unique representation

$$x = \sum_{k=1}^{\infty} x_k e^{(k)}, \text{ that is, } x = \lim_{n \to \infty} \sum_{k=1}^{n} x_k e^{(k)}.$$
Example 2.2 The set \( \omega \) is a Fréchet space with ([26, Example 11.3.1])

\[
(\omega, d_\omega)
\]

(2.3) \[ d_\omega(x, y) = \sum_{k=1}^{\infty} \frac{1}{2^k} \cdot \frac{|x_k - y_k|}{1 + |x_k - y_k|} \text{ for all } x, y \in \omega \]

and

(i) convergence in \((\omega, d_\omega)\) is equivalent to coordinatewise convergence

([26, Theorem 4.1.1 and Example 4.1.5]).

So \((\omega, d_\omega)\) is an FK space, which obviously has AK.

Remark 2.3 In view of (i), an FK space \(X\) is a

(ii) Fréchet sequence space with its metric stronger than the metric \(d_\omega\) of \(\omega\) on \(X\),

or equivalently,

(iii) a Fréchet sequence space which is continuously embedded in \(\omega\).
The $FK$ spaces $\ell(p)$ and $c_0(p)$

**Example 2.4** Let $p = (p_k)_{k=1}^\infty$ be a bounded sequence of positive reals with $M = \max\{1, \sup_k p_k\}$. Then

$$\ell(p) = \{x \in \omega : \sum_{k=1}^\infty |x_k|^{p_k} < \infty\}$$

and

$$c_0(p) = \{x \in \omega : \lim_{k \to \infty} |x_k|^{p_k} = 0\}$$

are $FK$ spaces with $AK$ with

$$d_{(p)}(x, y) = \left(\sum_{k=1}^\infty |x_k - y_k|^{p_k}\right)^{1/M} \quad ([15, \text{Theorem 1}])$$

and

$$d_{0,(p)}(x, y) = \sup_k |x_k - y_k|^{p_k/M} \quad ([18, \text{Theorem 2}]).$$
The classical $BK$ spaces

**Example 2.5 (a)** The sets $\ell_p$ ($1 \leq p < \infty$), $c_0$, $c$ and $\ell_\infty$ are $BK$ spaces with

$$\|x\|_p = \left( \sum_{k=1}^{\infty} |x_k|^p \right)^{1/p} \quad \text{and} \quad \|x\|_\infty = \sup_k |x_k|$$ ([26, Example 11.3.1]);

c$0$ is a closed subspace of $c$ and $c$ is a closed subspace of $\ell_\infty$ ([27, Corollary 4.2.4]); $\ell_p$ and $c_0$ have $AK$ ([27, Example 7.3.7]);

every sequence $x = (x_k)_{k=1}^{\infty} \in c$ has a unique representation

$$x = \xi \cdot e + \sum_{k=1}^{\infty} (x_k - \xi) e^{(k)} \quad \text{where} \quad \xi = \lim_{k \to \infty} x_k$$ ([17, Example 3.4.11]);

$\ell_\infty$ has no Schauder basis, since it is not separable ([17, Theorem 3.4.7, Problem 3.4.4]).
2.3 The dual spaces

If $X, Y \subset \omega$, then
\[ M(X, Y) = \{ a \in \omega : ax = (a_k x_k)_{k=1}^{\infty} \in Y \text{ for all } x \in X \} \]
is called **multiplier space of** $X$ **and** $Y$;

the sets
\[ X^\alpha = M(X, \ell_1), \quad X^\beta = M(X, cs) \text{ and } X^\gamma = M(X, cs) \]
are called the $\alpha$–, $\beta$– and $\gamma$–duals of $X$.

Properties of duals of $BK$ spaces

**Theorem 2.6** ([27, Theorem 4.3.15]) Let $X$ and $Y$ be $BK$ spaces. Then $M(X, Y)$ is a $BK$ space with
\[ \|z\| = \sup\{\|xz\| : \|x\| = 1\} \quad (z \in M(X, Y)); \]
in particular, the $X^\alpha$, $X^\beta$, and $X^\gamma$ are $BK$ spaces.
Theorem 2.6 does not extend to $FK$ spaces, in general

Example 2.7 The space $(\omega, d_\omega)$ is an $FK$ space (Example 2.2) and $\omega^\beta = \phi$, as we will see in Example 2.10 (i), but $\phi$ has no Fréchet topology ([27, 4.0.5]).

Relation between $X'$ and $X^\beta$

Theorem 2.8 (a) If $X \supset \phi$ is an $FK$ space with $AK$ then $X^\beta = X^\gamma$ ([27, Theorem 7.2.7 (iii)]).

(b) ([27, Theorem 7.2.9]) Let $X \supset \phi$ be an $FK$ space. Then $X^\beta \subset X'$ in the sense that each sequence $a \in X^\beta$ can be used to represent a function $f_a \in X'$ with $f_a(x) = \sum_{k=1}^{\infty} a_k x_k$ for all $x \in X$, and the map $T : X^\beta \rightarrow X'$ with $T(a) = f_a$ is linear and one to one. If $X$ has $AK$ then $T$ is an isomorphism.
The norm $\| \cdot \|_X^*$

Let $X \subset \omega$ be a normed sequence space, and $a \in \omega$. Then we write

$$\|a\|_X^* = \sup_{x \in S_X} \left| \sum_{k=1}^{\infty} a_k x_k \right|$$

provided the expression on the righthand side exists and is finite which is the case whenever $X$ is a $BK$ space and $a \in X^\beta$ by Theorem 2.8 (b).

Norm isomorphic, $\equiv$

If $X$ and $Y$ are norm isomorphic then we write

$$X \equiv Y.$$
The $\beta$–duals of $\ell(p)$ and $c_0(p)$

**Example 2.9** (a) If $p_k > 1$ and $q_k = p_k/(p_k - 1)$ for all $k$, then

$$(\ell(p))^\beta = \bigcup_{N>1} \left\{ a \in \omega : \sum_{k=1}^{\infty} \left| \frac{a_k}{N} \right| q_k < \infty \right\}$$

([16, Theorem 11]).

(b) If $p$ is any sequence of positive reals then,

$$(c_0(p))^\beta = \bigcup_{N>1} \left\{ a \in \omega : \sum_{k=1}^{\infty} |a_k| \cdot N^{-1/p_k} < \infty \right\}$$

([16, Theorem 6]).
The duals of the classical $BK$ spaces

**Example 2.10 ([26, Examples 6.4.2, 6.4.3 and 6.4.4])** We have

(i) $\omega^\beta = \phi$ and $\phi^\beta = \omega$

(ii) $\ell_1^* \equiv \ell_1^\beta = \ell_\infty$; $\ell_p^* \equiv \ell_p^\beta = \ell_q$ for $1 < p < \infty$ and $q = p/(p - 1)$

(iii) $c_0^* \equiv c_0^\beta = c^\beta = \ell_\infty = \ell_1$

(iv) $f \in c^*$ if and only if

$$f(x) = \chi f \cdot \lim_{k \to \infty} x_k + \sum_{k=1}^{\infty} a_k x_k$$

where $a = (f(e^{(k)}))_{k=1}^{\infty} \in \ell_1$ and

$$\chi f = f(e) - \sum_{k=1}^{\infty} f(e^{(k)})$$

and $\|f\| = |\chi f| + \|a\|_1$

(v) $\ell_\infty^*$ is not given by any sequence space ([26, Example 6.4.8]), but

(2.4) $\| \cdot \|_{\ell_\infty^*} = \| \cdot \|_1$ on $\ell_\infty^\beta$. 
2.4 General results

We list a few useful known results.

The relation between $\mathcal{B}(X, Y)$ and $(X, Y)$

Theorem 2.11 Let $X$ and $Y$ be $BK$ spaces.

(a) Then we have $(X, Y) \subset \mathcal{B}(X, Y)$; this means that if $A \in (X, Y)$, then $L_A \in \mathcal{B}(X, Y)$ where

$$L_A(x) = Ax \ (x \in X) \ ([27, \text{ Theorem 4.2.8}]).$$

(b) If $X$ has $AK$ then we have $\mathcal{B}(X, Y) \subset (X, Y)$; this means every $L \in \mathcal{B}(X, Y)$ is given by a matrix $A \in (X, Y)$ such that

$$L(x) = Ax \ (x \in X) \ ([12, \text{ Theorem 1.9}]).$$
The class \((X, \ell_\infty)\) ([20, Theorem 1.23 (b)])

**Theorem 2.12** Let \(X\) be a \(BK\) space. Then \(A \in (X, \ell_\infty)\) if and only if
\[
\|A\|_{(X, \ell_\infty)} = \sup_n \|A_n\|_X^* < \infty;
\]
moreover, if \(A \in (X, Y)\) then \(\|L_A\| = \|A\|_{(X, \ell_\infty)}\).

Improvement of mapping

**Theorem 2.13** (a) ([27, 8.3.6]) Let \(X\) be an \(FK\) space with \(\overline{\phi} = X, Y\) and \(Y_1\) be \(FK\) spaces with \(Y_1\) a closed subspace of \(Y\). Then we have
\[
A \in (X, Y_1) \text{ if and only if } A \in (X, Y) \text{ and } Ae^{(k)} \in Y_1 \text{ for all } k.
\]
(b) ([27, 8.3.7]) Let \(X\) be an \(FK\) space and \(X_1 = X \oplus e\). Then
\[
A \in (X_1, Y) \text{ if and only if } A \in (X, Y) \text{ and } Ae \in Y.
\]
Mapping properties of the transpose

Let $A^t$ denote the transpose of the matrix $A$.

**Theorem 2.14** ([27, Theorem 8.3.9]) Let $X$ and $Z$ be BK spaces with $AK$ and $Y = Z^\beta$. Then we have

(a) \((X^{\beta\beta}, Y) = (X, Y)\);

(b) \(A \in (X, Y)\) if and only if $A^t \in (Z, X^\beta)$. 

The classes \((X, Y)\) for \(X = \ell_\infty, c, c_0\)

**Example 2.15** We have

(a) \(A \in (\ell_\infty, \ell_\infty)\) if and only if

\[
(2.5) \quad \|A\|_{(\infty, \infty)} = \sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty \quad ([27, 8.4.5A]);
\]

if \(A \in (\ell_\infty, \ell_\infty)\) then

\[
(2.6) \quad \|L_A\| = \|A\|_{(\infty, \infty)}
\]

(b) \((c_0, \ell_\infty) = (c, \ell_\infty) = (\ell_\infty, \ell_\infty)\) \([27, 8.4.5A]\).

**Proof.** (a) Since \(\ell_\infty\) is a \(BK\) space by Example 2.5 (a), the statements follow from Theorem 2.12 and (2.4).

(b) Since \(c_0\) is a \(BK\) space with \(AK\) by Example 2.5 (a), and \(c_0^{\beta\beta} = \ell_1^\beta = \ell_\infty\) by Example 2.10 (iii) and (ii), Theorem 2.14 (a) yields \((c_0, \ell_\infty) = (\ell_\infty, \ell_\infty)\); this and \(c_0 \subset c \subset \ell_\infty\) yield \((c, \ell_\infty) = (\ell_\infty, \ell_\infty)\).
The class $\mathcal{B}(\ell_1, \ell_1)$

**Example 2.16 ([20, Theorem 2.27])** We have $\mathcal{B}(\ell_1, \ell_1) = (\ell_1, \ell_1)$ by Theorem 2.11 and Example 2.5 (b). Also $A \in (\ell_1, \ell_1)$ if and only if

$$\|A\|_{(1,1)} = \sup_k \sum_{n=1}^{\infty} |a_{nk}| < \infty$$

moreover, if $L \in \mathcal{B}(\ell_1, \ell_1)$ then

$$\|L\| = \|A\|_{(1,1)}.$$  

**Proof.** We apply Theorem 2.14 (b) with $X = \ell_1$ and $Z = c_0$, $BK$ spaces with $AK$ by Example 2.5, and $Y = c^\beta_0 = \ell_1$ by Example 2.10 (iii) to obtain $A \in (\ell_1, \ell_1)$ if and only if $A^t \in (\ell_\infty, \ell_\infty)$, and this is the case by (2.5) if and only if

$$\|A^t\|_{(\infty,\infty)} = \sup_n \sum_{k=1}^{\infty} |a_{kn}| = \|A\|_{(1,1)} < \infty.$$
Furthermore, if $L \in (\ell_1, \ell_1)$, then

$$
\|L(x)\|_1 = \|Ax\|_1 = \sum_{n=1}^{\infty} |A_n x| = \sum_{n=1}^{\infty} \left| \sum_{k=1}^{\infty} a_{nk} x_k \right|
$$

$$
\leq \sum_{k=1}^{\infty} |x_k| \left( \sum_{n=1}^{\infty} |a_{nk}| \right) \leq \|A\|_{(1,1)} \cdot \|x\|_1
$$

implies

(2.9) \quad \|L\| \leq \|A\|_{(1,1)}.

We also have for $e^{(k)} \in S_{\ell_1} \ (k \in \mathbb{N})$

$$
\|L(e^{(k)})\|_1 = \|A e^{(k)}\|_1 = \sum_{n=1}^{\infty} |A_n e^{(k)}| = \sum_{n=1}^{\infty} |a_{nk}| \leq \|L\|,
$$

hence $\|A\|_{(1,1)} \leq \|L\|$. This and (2.9) imply (2.8).
Regular matrices, the Silverman–Toeplitz theorem

A matrix $A \in (c, c)$ is said to be regular if $\lim_{n \to \infty} A_n x = \lim_{k \to \infty} x_k$ for all $x \in c$.

Theorem 2.17 (Silverman–Toeplitz, 1911)
(a) ([27, Theorem 1.3.6]) We have $A \in (c, c)$ if and only if

(i) $\|A\|_{(\infty, \infty)} = \sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty$ ((2.5)),

(ii) $\alpha_k = \lim_{n \to \infty} a_{nk}$ exists for each $k \in \mathbb{N}$,

(iii) $\tilde{\alpha} = \lim_{n \to \infty} \sum_{k=1}^{\infty} a_{nk}$ exists.
(b) Let $A \in (c, c)$ and $x \in c$. Then we have

\[
\lim_{n \to \infty} A_n x = \left( \tilde{\alpha} - \sum_{k=1}^{\infty} \alpha_k \right) \cdot \lim_{k \to \infty} x_k + \sum_{k=1}^{\infty} \alpha_k x_k.
\]

(c) A matrix $A$ is regular if and only if (i) holds and

\[
(ii') \lim_{n \to \infty} a_{nk} = 0 \text{ for each } k \in \mathbb{N},
\]

\[
(iii') \lim_{n \to \infty} \sum_{k=1}^{\infty} a_{nk} = 1.
\]

Proof. Part (a) follows from Theorem 2.13 (a) and (b) and Example 2.15. (b) This is elementary. Part (c) follows from Parts (a) and (b).
Schur’s theorem; proof by the method of the gliding hump

**Theorem 2.18 (Schur)** We have

(a) ([27, Theorem 1.7.18]) \( A \in (\ell_\infty, c) \) if and only if

\[
\sum_{k=1}^{\infty} |a_{nk}| \text{ converges uniformly in } n
\]

and

\[
\lim_{n \to \infty} a_{nk} = \alpha_k \text{ exists for each } k \in \mathbb{N};
\]

(b) ([27, Theorem 1.7.19]) \( A \in (\ell_\infty, c_0) \) if and only if condition (2.11) holds and

\[
\lim_{n \to \infty} a_{nk} = 0 \text{ for each } k \in \mathbb{N}.
\]

**Remark 2.19** The conditions in (2.11) and (2.13) are equivalent to

\[
\lim_{n \to \infty} \sum_{k=1}^{\infty} |a_{nk}| = 0 ([25, 21. (21.1)]).
\]
Applications

Example 2.20 (Steinhaus) A regular matrix cannot sum all bounded sequences.

Proof. If there were a matrix regular $A \in (\ell_\infty, c)$, then it would follow from the conditions in (2.11) of Theorem 2.18 and in (ii’) and (iii’) in Part (c) of Theorem 2.17 that

$$1 = \lim_{n \to \infty} \sum_{k=1}^{\infty} a_{nk} = \sum_{k=1}^{\infty} \lim_{n \to \infty} a_{nk} = 0,$$

a contradiction. □

Example 2.21 Schur’s theorem can be applied to show that weak and strong convergence coincide in $\ell_1$. 

30
Proof of Example 2.21

We assume that the sequence \((x^{(n)})_{n=1}^{\infty}\) is weakly convergent to \(x\) in \(\ell_1\), that is,
\[
f(x^{(n)}) - f(x) \to 0 \quad (n \to \infty)
\]
for each \(f \in \ell_1^*\).

To every \(f \in \ell_1^*\) there corresponds a sequence \(a \in \ell_\infty\) (Example 2.10 (ii)) such that
\[
f(y) = \sum_{k=1}^{\infty} a_k y_k \quad \text{for all } y \in \ell_1.
\]

We define the matrix \(B = (b_{nk})_{n,k=1}^{\infty}\) by \(b_{nk} = x^{(n)}_k - x_k\) \((n, k = 1, 2, \ldots)\).

Then we have for all \(a \in \ell_\infty\)
\[
f(x^{(n)}) - f(x) = f(x^{(n)} - x) = \sum_{k=1}^{\infty} a_k (x^{(n)}_k - x_k)
\]
\[
\sum_{k=1}^{\infty} b_{nk} a_k \to 0 \quad (n \to \infty),
\]
that is, \( B \in (\ell_\infty, c_0) \). It follows from Theorem 2.18 (b), that \( \sum_{k=1}^{\infty} |b_{nk}| \) converges uniformly in \( n \) and \( \lim_{n \to \infty} b_{nk} = 0 \) for each \( k \). Thus we have

\[
\|x^{(n)} - x\|_1 = \sum_{k=1}^{\infty} |x_{k}^{(n)} - x_k| = \sum_{k=1}^{\infty} |b_{nk}| \to 0 \quad (n \to \infty).
\]
3 Compact Operators

The $\textit{Hausdorff measure of noncompactness}$ is most effective in determining conditions for a linear operator to be compact.

Measures of noncompactness are also very useful tools in the theory of operator equations in Banach spaces, in particular in fixed point theory.

The first measure of noncompactness, denoted by $\alpha$, was defined and studied by Kuratowski [13] in 1930. Later, in 1955, Darbo [6] used the function $\alpha$ to prove a generalisation of Schauder’s fixed point theorem to noncompact operators.


The general theory of measures of noncompactness can be found in [1, 2].
Notations

We write $\mathcal{M}_X$ for the class of all bounded subsets of a metric space $(X, d)$.

Let $X$ and $Y$ be infinite–dimensional complex Banach spaces. A linear operator $L : X \to Y$ is said to be compact if

- $D_L = X$ for the domain $D_L$ of $L$
- for every bounded sequence $(x_n)$ in $X$, the sequence $(L(x_n))$ has a convergent subsequence.

We write $C(X, Y)$ for the class of all compact operators from $X$ into $Y$.

A norm $\| \cdot \|$ on a sequence space is said to be monotone, if $x, x' \in X$ with $|x_k| \leq |x'_k|$ for all $k$ implies $\|x\| \leq \|x'\|$. 
3.1 The Hausdorff measure of noncompactness

We introduce the notion of a measure of noncompactness.

**Definition 3.1** ([1, Definition II.1.1]) Let $X$ be a metric space. A map

$$\mu : \mathcal{M}_X \rightarrow [0, \infty)$$

is said to be a **measure of noncompactness on $X$ (MNC)** if it satisfies the following properties

(MNC.1) $\mu(Q) = 0$ if and only if $Q$ is relatively compact \hspace{1cm} (Regularity)

(MNC.2) $\mu(Q) = \mu(\overline{Q})$ for all $Q \in \mathcal{M}_X$ \hspace{1cm} (Invariance under closure)

(MNC.3) $\mu(Q_1 \cup Q_2) = \max\{\mu(Q_1), \mu(Q_2)\}$ for all $Q_1, Q_2 \in \mathcal{M}_X$ \hspace{1cm} (Semi-additivity)
Properties of $MNC$’s in metric spaces

**Theorem 3.2** ([20, Lemma 2.11] or [1, (1),(2),(3), p. 19])

Let $\mu$ be a $MNC$ in a metric space $X$.

Then we have for all $Q, Q_1, Q_2 \in \mathcal{M}_X$

- $(MNC.1')$ \hspace{1cm} $Q_1 \subset Q_2$ implies $\mu(Q_1) \leq \mu(Q_2)$ \hspace{1cm} (Monotonicity)

- $(MNC.2')$ \hspace{1cm} $\mu(Q_1 \cap Q_2) \leq \min\{\mu(Q_1), \mu(Q_2)\}$

- $(MNC.3')$ \hspace{1cm} $\mu(Q) = 0$ for every finite set $Q$ \hspace{1cm} (Non–singularity)
Properties of $MNC$’s in normed spaces

**Theorem 3.3** ([20, Theorem 2.12] or [1, (6),(5),(7), p. 19])

Let $\mu$ be a $MNC$ in a normed space $X$.

Then we have for all $Q, Q_1, Q_2 \in \mathcal{M}_X$ and all scalars $\lambda$

\[(MNC.4') \quad \mu(Q_1 + Q_2) \leq \mu(Q_1) + \mu(Q_2)\]

(Algebraic semi–additivity)

\[(MNC.5') \quad \mu(\lambda Q_1) = |\lambda|\mu(Q_1)\] (Homogeneity)

\[(MNC.6') \quad \mu(x + Q) = \mu(Q)\] (Translation invariance)
Hausdorff MNC of bounded sets

Definition 3.4 ([20, Definition 2.10] or [1, Definition II.2.1])

Let \((X, d)\) be a metric space and \(Q \in \mathcal{M}_X\).

The Hausdorff measure of noncompactness (HMNC) of \(Q\) is defined by

\[
\chi(Q) = \inf \left\{ \epsilon > 0 : Q \subset \bigcup_{i=1}^{n} B(r_i, x_i); \quad r_i < \epsilon, \ x_i \in X \right\}.
\]
Invariance of the HMNC under the passage to the convex hull

Proposition 3.5 ([20, Theorem 2.12 (2.31)] or [1, Theorem II.2.4])
Let $X$ be a normed space. Then we have for all $Q \in \mathcal{M}_X$

$$\chi(\text{co}(Q)) = \chi(Q) \text{ where } \text{co}(Q) \text{ is the convex hull of } Q.$$  

HMNC of the unit ball in a normed space

Theorem 3.6 ([20, Theorem 2.14]) or [1, Theorem II.2.5])
Let $X$ be an infinite dimensional normed space. Then we have

$$\chi(B_X) = 1.$$  

(3.1)
HMNC of bounded sets in $BK$ spaces

**Theorem 3.7 ([3, Theorem 3.4])** (a) Let $X$ be a monotone $BK$ space with $AK$ and $P_n : X \to X$ be the projectors onto the linear span of \{e(1), e(2), \ldots , e(n)\} for $n = 1, 2, \ldots$. Then

\[
\chi(Q) = \lim_{n \to \infty} \left( \sup_{x \in Q} \|(I - P_n)(x)\| \right) \quad \text{for all } Q \in \mathcal{M}_X.
\]

(b) Let $P_n : c \to c$ be the projectors onto the linear span of \{e, e(1), \ldots , e(n)\} for $n = 1, 2, \ldots$. Then

\[
\frac{1}{2} \cdot \lim_{n \to \infty} \left( \sup_{x \in Q} \|(I - P_n)(x)\| \right) \leq \chi(Q) \leq \lim_{n \to \infty} \left( \sup_{x \in Q} \|(I - P_n)(x)\| \right) \quad \text{for all } Q \in \mathcal{M}_X.
\]
Definition 3.8 ([20, Definition 2.24])

Let $X$ and $Y$ be Banach spaces and $\chi_1$ and $\chi_2$ be HMNC’s on $X$ and $Y$. Then the operator $L : X \to Y$ is called $(\chi_1, \chi_2)$-bounded if

- $L(Q) \in \mathcal{M}_Y$ for every $Q \in \mathcal{M}_X$
- there exists a constant $C > 0$ such that

\[
(3.4) \quad \chi_2(L(Q)) \leq C \cdot \chi_1(Q) \quad \text{for every } Q \in \mathcal{M}_X.
\]

If an operator $L$ is $(\chi_1, \chi_2)$-bounded then the number

\[
\|L\|_{(\chi_1, \chi_2)} = \inf\{C > 0 : (3.4) \text{ holds}\}
\]

is called the $(\chi_1, \chi_2)$-measure of noncompactness of $L$.

If $\chi_1 = \chi_2 = \chi$, then we write $\|L\|_\chi = \|L\|_{(\chi, \chi)}$. 

41
Properties of the HMNC of operators

**Theorem 3.9 ([20, Theorem 2.25])** Let $X$ and $Y$ be Banach spaces and $L \in \mathcal{B}(X, Y)$. Then

$$\|L\|_\chi = \chi(L(S_X)) = \chi(L(B_X)).$$

**Theorem 3.10 ([20, Corollary 2.26])** Let $X$, $Y$, and $Z$ be Banach spaces, $L \in \mathcal{B}(X, Y)$ and $\tilde{L} \in \mathcal{B}(Y, Z)$. Then $\| \cdot \|_\chi$ is a seminorm on $\mathcal{B}(X, Y)$ and

$$\|L\|_\chi = 0 \text{ if and only if } L \in \mathcal{C}(X, Y),$$

$$\|L\|_\chi \leq \|L\|,$$

$$\|L + K\|_\chi = \|L\|_\chi \text{ for each } K \in \mathcal{C}(X, Y),$$

$$\|\tilde{L} \circ L\|_\chi \leq \|\tilde{L}\|_\chi \cdot \|L\|_\chi.$$
Example 3.11 (Goldenštein, Gohberg and Markus) ([20, Theorem 2.28]) Let \( L \in \mathcal{B}(\ell_1, \ell_1) \). Then we have by Example 2.16, (2.7), (2.8), (3.2) and (3.5)

\[
\|L\|_\chi = \|L_A\|_\chi = \lim_{m \to \infty} \left( \sup_{k} \sum_{n=m}^{\infty} |a_{nk}| \right).
\]

Example 3.12 (Goldenštein, Gohberg and Markus) ([20, Corollary 2.29]) Let \( L \in \mathcal{B}(\ell_1, \ell_1) \). Then we have by (3.6) and (3.7), \( L \in \mathcal{C}(\ell_1, \ell_1) \) if and only if

\[
\lim_{m \to \infty} \left( \sup_{k} \sum_{n=m}^{\infty} |a_{nk}| \right) = 0.
\]
Compact matrix operators between the classical sequence spaces

Remark 3.13 (a) It is clear from (3.8) that $I \in B(\ell_1, \ell_1) \setminus C(\ell_1, \ell_1)$.

(b) The necessary and sufficient conditions for an operator given by a matrix $A \in (X, Y)$ to be compact can be found in [23, (a)--(b), p. 85] when

$$X = \ell_1, \ell_\infty, c_0, c, \quad Y = \ell_r \text{ for } 1 \leq r < \infty;$$

$$X = \ell_p \text{ for } 1 < p \leq \infty, \quad Y = \ell_1, \ell_\infty;$$

$$X = c_0, \quad Y = \ell_\infty.$$

(c) The class $C(\ell_1, \ell_\infty)$ cannot be determined by the use of the Hausdorff measure of noncompactness. The characterisation of the class $C(\ell_1, \ell_\infty)$ is established in [23, Theorem 5].
An estimate for $\| L \|_\chi$ when $L \in \mathcal{B}(X, c)$

**Theorem 3.14** ([3, Theorem 3.5]) Let $X$ be a $BK$ space with $AK$.

Then every $L \in \mathcal{B}(X, c)$ can be represented by a matrix $A = (a_{nk})_{n,k=1}^{\infty} \in (X, c)$ such that $L(x) = Ax$ for all $x \in X$ (Theorem 2.11 (b)).

The Hausdorff measure of noncompactness of $L$ satisfies

$$\frac{1}{2} \cdot \lim_{r \to \infty} \left( \sup_{n \geq r} \| A_n - \alpha \|_X^* \right) \leq \| L \|_\chi \leq \lim_{r \to \infty} \left( \sup_{n \geq r} \| A_n - \alpha \|_X^* \right),$$

where

$$\alpha_k = \lim_{k \to \infty} a_{nk} \text{ for every } k \in \mathbb{N} \text{ and } (\alpha_k)_{k=1}^{\infty} \in X^\beta.$$
**Representation of** \( L \in \mathcal{B}(c, c) \)

**Lemma 3.15** Every \( L \in \mathcal{B}(c, c) \) is given by a matrix \( B = (b_{nk})_{n=1,k=0}^{\infty} \) such that

\[
L(x) = (b_{n0} \cdot \xi + \sum_{k=1}^{\infty} b_{nk} x_k)_{n=1}^{\infty} \text{ for all } x \in c, \text{ where } \xi = \lim_{k \to \infty} x_k,
\]

\[
(3.11) \quad \beta_k = \lim_{n \to \infty} b_{nk} \text{ exists for all } k \geq 1, \quad \beta = \lim_{n \to \infty} \sum_{k=0}^{\infty} b_{nk} \text{ exists},
\]

\[
\|L\| = \sup_{n \in \mathbb{N}} \left( \sum_{k=0}^{\infty} |b_{nk}| \right).
\]

Furthermore, we have

\[
\eta = \lim_{n \to \infty} (L(x))_n = (\beta - \sum_{k=1}^{\infty} \beta_k) \xi + \sum_{k=1}^{\infty} \beta_k x_k \text{ for all } x \in c.
\]
A formula for $\|L\|_\chi$ when $L \in \mathcal{B}(c, c)$

**Theorem 3.16** ([19, Theorem 1]) *If* $L \in \mathcal{B}(c, c)$, *then* we have

$$
\frac{1}{2} \cdot \lim_{n \to \infty} \left( \left| b_{n0} - \beta + \sum_{k=1}^{\infty} \beta_k \right| + \sum_{k=1}^{\infty} |b_{nk} - \beta_k| \right)
$$

$$
\leq \|L\|_\chi \leq \lim_{n \to \infty} \left( \left| b_{n0} - \beta + \sum_{k=1}^{\infty} \beta_k \right| + \sum_{k=1}^{\infty} |b_{nk} - \beta_k| \right),
$$

*where* $b_{nk}$ ($n \in \mathbb{N}; k \in \mathbb{N}_0$) *are the entries of the matrix* $B$ *that represents* $L$ *by Lemma 3.15,*

*and* $\beta$ *and* $\beta_k$ ($k \in \mathbb{N}$) *are given by* (3.11).
Characterisation of $C(X, Y)$ when $X, Y = c_0, c$

**Corollary 3.17** Let $L \in \mathcal{B}(X, Y)$. Then the necessary and sufficient conditions for $L \in C(X, Y)$ can be read from the table

$$
\begin{array}{|c|c|c|}
\hline
\text{From} & c_0 & c \\
\hline
\text{To} & & \\
\hline
c_0 & 1. & 2. \\
c & 3. & 4. \\
\hline
\end{array}
$$

where

1. $\lim_{r \to \infty} (\sup_{n \geq r} \sum_{k=1}^{\infty} |a_{nk}|) = 0$;
2. $\lim_{n \to \infty} (\sum_{k=0}^{\infty} |b_{nk}|) = 0$;
3. $\lim_{r \to \infty} (\sup_{n \geq r} \sum_{k=1}^{\infty} |a_{nk} - \alpha_k|) = 0$;
4. $\lim_{n \to \infty} \left( |b_{n0} - \beta + \sum_{k=1}^{\infty} \beta_k| + \sum_{k=1}^{\infty} |b_{nk} - \beta_k| \right) = 0$. 


Compact matrix operators in \((c, c_0)\) and \((c, c)\)

**Corollary 3.18** (a) Let \(A \in (c, c_0)\). Then we have \(L_A \in \mathcal{C}(c, c_0)\) if and only if

\[
\lim_{n \to \infty} \sum_{k=1}^{\infty} |a_{nk}| = 0.
\]

(b) Let \(A \in (c, c)\). Then we have \(L_A \in \mathcal{C}(c, c)\) if and only if

\[
\lim_{n \to \infty} \left( \left| \sum_{k=1}^{\infty} \alpha_k - \alpha \right| + \sum_{k=1}^{\infty} |a_{nk} - \alpha_k| \right) = 0
\]

with \(\alpha_k\) (\(k \in \mathbb{N}\)) from (3.10) or (ii) in Theorem 2.17, and

\[
\alpha = \lim_{n \to \infty} \sum_{k=1}^{\infty} a_{nk} \quad (\text{iii) in Theorem 2.17}).
\]
Regular operators

An operator \( L \in \mathcal{B}(c, c) \) is said to be regular, if
\[
\lim_{n \to \infty} L_n(x) = \xi \text{ for all } x \in c, \text{ where } \xi = \lim_{k \to \infty} x_k.
\]

**Corollary 3.19** ([5, Corollary 3]) Let \( L \in \mathcal{B}(c, c) \) be regular. Then we have \( L \in \mathcal{C}(c, c) \) if and only if
\[
(3.12) \quad \lim_{n \to \infty} \left( |b_{n0} - 1| + \sum_{k=1}^{\infty} |b_{nk}| \right) = 0
\]
with \( b_{nk} \) \((n \in \mathbb{N}; k \in \mathbb{N}_0)\) from Theorem 3.16.

**Remark 3.20** If \( A \) is a regular matrix then \( L_A \) cannot be compact, since we have \( b_{n0} = 0 \) for all \( n \in \mathbb{N} \) and
\[
1 + \sum_{k=1}^{\infty} |a_{nk}| \geq 1 \neq 0 \text{ for all } n,
\]
and so (3.12) in Corollary 3.19 cannot hold.
4 Fredholm Operators on $c_0$

Here we apply our previous results to establish sufficient conditions for a matrix operator in $c_0$ to be a Fredholm operator. The presented results are the special cases $T = I$ of those in [7]

If we denote the set of finite rank operators by $\mathcal{F}(X, Y)$, and suppose that $X$ is a normed space and $Y$ is a Banach space, then it is well known that $\mathcal{F}(X, Y) \subset \mathcal{C}(X, Y)$ ([22], p.111). In particular, if $X$ is a Banach space and $Y$ is a Hilbert space, then the set of compact operators is the closure of the set of finite rank operators, that is, $\overline{\mathcal{F}(X, Y)} = \mathcal{C}(X, Y)$ ([22], p.111). But, here, we deal with Banach spaces.

The concept of the finite dimensional case connects compact and Fredholm operators. Here, we will use a result based on the relation between them and compact operators.
Fredholm operators

Definition 4.1 Let $X$ and $Y$ be Banach spaces and $L \in \mathcal{B}(X, Y)$. We denote the null and the range spaces of $L$ by $N(L)$ and $R(L)$. Then $L$ is said to be a **Fredholm operator** if the following conditions hold:

1. $N(L)$ is finite dimensional;
2. $Y/R(L)$ is finite dimensional.

The set of Fredholm operators from $X$ to $Y$ is denoted by $\Phi(X, Y)$; we write $\Phi(X) = \Phi(X, X)$.

Remark 4.2 The range of a Fredholm operator is closed.

Known result

Theorem 4.3 ([24, p.106]) Let $X$ be a Banach space and $L \in \mathcal{B}(X) = \mathcal{B}(X, X)$, then $I - L \in \Phi(X)$ where $I$ is the identity operator on $X$. 
A sufficient condition for $\mathcal{B}(X) \subset \Phi(X)$

**Theorem 4.4** Let $L \in \mathcal{B}(c_0)$. If

$$
(4.1) \quad \lim_{r \to \infty} \left[ \sup_{n \geq r} \left( |1 - a_{nn}| + \sum_{k=1, k \neq n}^{\infty} |a_{nk}| \right) \right] = 0,
$$

then we have $L \in \Phi(c_0)$.

**Proof.** We put $C = I - A$ where $A$ is the matrix with $Ax = L(x)$ for all $x \in c_0$ (Theorem 2.11 (b)). Then $L_C \in \mathcal{C}(c_0, c_0)$ by 1. in Corollary 3.17 if and only if

$$
\lim_{r \to \infty} \left( \sup_{n \geq r} \sum_{k=1}^{\infty} |c_{nk}| \right) = 0
$$

which is (4.1). Now the statement follows with Theorem 4.3. ■
5 The Spaces of Strongly Summable and Bounded Sequences

Let $1 \leq p < \infty$ throughout

\[
w_0^p = \left\{ x \in \omega : \lim_{n \to \infty} \left( \frac{1}{n} \sum_{k=1}^{n} |x_k|^p \right) = 0 \right\},
\]

\[
w^p = \left\{ x \in \omega : \lim_{n \to \infty} \left( \frac{1}{n} \sum_{k=1}^{n} |x_k - \xi|^p \right) = 0 \text{ for some } \xi \in \mathbb{C} \right\},
\]

\[
w_\infty^p = \left\{ x \in \omega : \sup_n \left( \frac{1}{n} \sum_{k=1}^{n} |x_k|^p \right) < \infty \right\}.
\]
Strong limit

If $x \in w^p$ then the strong $C_1$–limit of $x$ is the number $\xi \in \mathbb{C}$ with

$$
\lim_{n \to \infty} \left( \frac{1}{n} \sum_{k=1}^{n} |x_k - \xi|^p \right) = 0.
$$

Sectional and block norms

We define the sectional $\| \cdot \|_s$ and block norms $\| \cdot \|_b$ by

$$
\|x\|_s = \sup_n \left( \frac{1}{n} \sum_{k=1}^{n} |x_k|^p \right)^{1/p}
$$

and

$$
\|x\|_b = \sup_{\nu} \left( \frac{1}{2^{\nu+1}} \sum_{k=2^\nu}^{2^{\nu+1}-1} |x_k|^p \right)^{1/p}.
$$
Topological properties of $w^p_0$, $w^p$ and $w^p_\infty$

**Theorem 5.1** (a) The strong limit is unique for each $x \in w^p$ ([14]).

(b) We have $w^p_0 \subset w^p \subset w^p_\infty$; $\| \cdot \|_s$ and $\| \cdot \|_b$ are equivalent norms on $w^p_0$, $w^p$ and $w^p_\infty$ ([21, Proposition 5.3]);

(c) $w^p_0$, $w^p$ and $w^p_\infty$ are $BK$ spaces with $\| \cdot \|_b$ ([21, Proposition 5.3]);

$w^p_0$ is a closed subspace of $w^p$, and $w^p$ is a closed subspace of $w^p_\infty$ (Part (b) and [27, Corollary 4.2.4]);

$w^p_0$ has $AK$ ([21, Remark 5.4 (b)]); every sequence $x = (x_k)_{k=1}^\infty \in w^p$ has a unique representation ([14])

\[
x = \xi \cdot e + \sum_{k=1}^\infty (x_k - \xi)e^{(k)} \quad \text{where } \xi \text{ is the strong limit of } x;
\]

$w^p_\infty$ has no Schauder basis ([21, Remark 5.4 (a)]).
Notations for the duals of $w_0^p$, $w^p$ and $w_\infty^p$

We write

$$\|a\|_{\mathcal{M}_p} = \begin{cases} 
\sum_{\nu=0}^{\infty} 2^\nu \max_{2^\nu \leq k \leq 2^{\nu+1} - 1} |a_k| & (p = 1) \\
\sum_{\nu=0}^{\infty} 2^{\nu/p} \left( \sum_{k=2^\nu}^{2^{\nu+1}-1} |a_k|^q \right)^{1/q} & (1 < p < \infty)
\end{cases}.$$ 

and

$$\mathcal{M}_p = \left\{ a \in \omega : \|a\|_{\mathcal{M}_p} < \infty \right\}.$$
The duals of $w^p_0$, $w^p$ and $w^p_\infty$

**Theorem 5.2** Let $\dagger$ denote any of the symbols $\alpha$, $\beta$ and $\gamma$. Then

(a) $(w^p_0)^\dagger = (w^p)^\dagger = (w^p_\infty)^\dagger = \mathcal{M}_p$ ([21, Theorem 5.5 (a)]);

(b) $(w^p_0)^* \equiv \mathcal{M}_p$ ([21, Theorem 5.5 (b)]);

(c) ([14]) $f \in (w^p)^*$ if and only if

$$f(x) = \chi_f \cdot \xi + \sum_{k=1}^{\infty} a_k x_k \text{ where } a = \left( f(e^{(k)}) \right)_{k=1}^{\infty} \in \mathcal{M}_p,$$

$\xi$ is the strong limit of $x$, and

$$\chi_f = f(e) - \sum_{k=1}^{\infty} f(e^{(k)}) \text{, and } \|f\| = |\chi_f| + \|a\|_{\mathcal{M}_p};$$

(d) $(w^p_\infty)^*$ is not given by any sequence space, but

$$\| \cdot \|_{w^p_\infty}^* = \| \cdot \|_{\mathcal{M}_p} \text{ on } (w^p_\infty)^\beta$$

([21, Remark 5.6 and Theorem 5.5 (a)]).
The second duals of $w_0^p$, $w^p$ and $w_\infty^p$

**Theorem 5.3 ([21, Theorem 5.7])**
The set $\mathcal{M}_p$ is a $BK$ space with $AK$ with $\| \cdot \|_{\mathcal{M}_p}$.

**Theorem 5.4 (a) ([21, Theorem 5.8 (a)])**
Let $\dagger$ denote any of the symbols $\alpha$, $\beta$ and $\gamma$.

Then we have

$$(w_0^p)^{\dagger\dagger} = (w^p)^{\dagger\dagger} = (w_\infty^p)^{\dagger\dagger} = w_\infty^p.$$  

(b) ([21, Theorem 5.8 (b)])
The continuous dual $(\mathcal{M}_p)^*$ of $\mathcal{M}_p$ is norm isomorphic with $(w_\infty^p, \| \cdot \|_b)$.  

59
Matrix transformations

**Theorem 5.5** ([4, Theorem 2.4]) *The necessary and sufficient conditions for* $A \in (X, Y)$ *when*

$$X \in \{w_0^p, w^p, w^p_\infty\} \text{ and } Y \in \{\ell_\infty, c, c_0\}$$

*can be read from the following table*

<table>
<thead>
<tr>
<th>From</th>
<th>$w^p_\infty$</th>
<th>$w^p_0$</th>
<th>$w^p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\ell_\infty$</td>
<td>1.</td>
<td>1.</td>
<td>1.</td>
</tr>
<tr>
<td>$c_0$</td>
<td>2.</td>
<td>3.</td>
<td>4.</td>
</tr>
<tr>
<td>$c$</td>
<td>5.</td>
<td>6.</td>
<td>7.</td>
</tr>
</tbody>
</table>

*where*
1. \((1.1)\) \(\sup_n \|A_n\|_{\mathcal{M}_p} < \infty\)

2. \((2.1)\) \(\lim_{n \to \infty} \|A_n\|_{\mathcal{M}_p} = 0\)

3. \((1.1)\) and \((3.1)\) \(\lim_{n \to \infty} a_{nk} = 0\) for all \(k\)

4. \((1.1)\), \((3.1)\) and \((4.1)\) \(\lim_{n \to \infty} \sum_{k=1}^{\infty} a_{nk} = 0\)

5. \((5.1)\) \(\alpha_k = \lim_{n \to \infty} a_{nk}\) exists for all \(k\),
   \(5.2\) \((\alpha_k)_{k=1}^{\infty}, A_n \in \mathcal{M}_p\) for all \(n\),
   \(5.3\) \(\lim_{n \to \infty} \|A_n - (\alpha_k)_{k=1}^{\infty}\|_{\mathcal{M}_p} = 0\)

6. \((1.1)\) and \((5.1)\)

7. \((1.1), (5.1)\) and \((7.1)\) \(\alpha = \lim_{n \to \infty} \sum_{k=1}^{\infty} a_{nk}\) exists.

**Remark 5.6** The conditions for \(A \in (w_p^\infty, c_0)\) and \(A \in (w_p^\infty, c)\) can be replaced by

2’. \((3.1)\) and \((2.1')\) \(\|A_n\|_{\mathcal{M}_p}\) converges uniformly in \(n\)

5’. \((2.1')\) and \((5.1)\).
An estimate for $\|L\|_\chi$ when $L \in \mathcal{B}(w_0^p, c)$

We obtain as an immediate consequence of Theorem 3.14:

**Corollary 5.7** ([3, Corollary 3.6]) Let $L \in \mathcal{B}(w_0^p, c)$.

Then we have

$$\frac{1}{2} \cdot \lim_{r \to \infty} \left( \sup_{n \geq r} \|A_n - \alpha\|_{\mathcal{M}_p} \right) \leq \|L\|_\chi \leq \lim_{r \to \infty} \left( \sup_{n \geq r} \|A_n - \alpha\|_{\mathcal{M}_p} \right).$$
Representation of $L \in \mathcal{B}(X, c_0)$ and equation for $\|L\|_X$.

**Corollary 5.8** ([3, Corollary 3.7]) Let $X$ be a $BK$ space with $AK$.

Then every operator $L \in \mathcal{B}(X, c_0)$ can be represented by an infinite complex matrix $A \in (X, c_0)$ such that $L(x) = Ax$ for all $x \in X$; also

$$\|L\|_X = \lim_{r \to \infty} \left( \sup_{n \geq r} \|A_n\|_X^* \right).$$

In particular, if $L \in \mathcal{B}(w_0^p, c_0)$, then we have ([3, Corollary 3.8])

$$\|L\|_X = \lim_{r \to \infty} \left( \sup_{n \geq r} \|A_n\|_{M_p} \right).$$
Representation of $L \in \mathcal{B}(w^p, c)$

**Theorem 5.9 ([3, Theorem 3.9])**  
(a) Every $L \in \mathcal{B}(w^p, c)$ is given by a matrix $B = (b_{nk})_{n=1,k=0}^\infty$ such that

$$L(x) = (b_{n0}\xi + \sum_{k=1}^{\infty} b_{nk}x_k)_{n=1}^\infty$$

for all $x \in w^p$, where $\xi \in \mathbb{C}$ is the strong $C_1$–limit of $x$,

$$\beta_k = \lim_{n \to \infty} b_{nk} \text{ exists for all } k \geq 1, \quad \beta = \lim_{n \to \infty} \sum_{k=0}^{\infty} b_{nk} \text{ exists},$$

(5.4)  

$$\|L\| = \sup_{n \in \mathbb{N}} \left( |b_{n0}| + \|B_n\|_{\mathcal{M}_p} \right).$$

Furthermore, we have

$$\eta = \lim_{n \to \infty} (L(x))_n = \xi\beta + \sum_{k=1}^{\infty} \beta_k(x_k - \xi)$$

$$= (\beta - \sum_{k=1}^{\infty} \beta_k) \xi + \sum_{k=1}^{\infty} \beta_k x_k \text{ for all } x \in w^p.$$
Estimate for $\|L\|_\chi$ when $L \in B(w^p, c)$

(b) If $L \in B(w^p, c)$, then we have

$$\frac{1}{2} \cdot \lim_{n \to \infty} \left( \left| b_{n0} - \beta + \sum_{k=1}^{\infty} \beta_k \right| + \left\| (b_{nk} - \beta_k)_{k=1}^{\infty} \right\|_{\mathcal{M}_p} \right)$$

$$\leq \|L\|_\chi \leq \lim_{n \to \infty} \left( \left| b_{n0} - \beta + \sum_{k=1}^{\infty} \beta_k \right| + \left\| (b_{nk} - \beta_k)_{k=1}^{\infty} \right\|_{\mathcal{M}_p} \right),$$

where $b_{nk}$ ($n \in \mathbb{N}; k \in \mathbb{N}_0$) are the entries of the matrix $B$ that represents $L$ by Part (a), and $\beta$ and $\beta_k$ ($k \in \mathbb{N}_0$) are given by and (5.4).

We obtain the following corollaries from Theorem 5.9.
Corollary 5.10 ([3, Corollary 3.10]) Let $A \in (w^p, c)$. Then we have
\[
\frac{1}{2} \lim_{n \to \infty} \left( |\sum_{k=1}^{\infty} \alpha_k - \alpha| + \|A_n - (\alpha_k)_{k=1}^{\infty} \| \right) \leq \|A\|_\chi
\]
\[
\leq \lim_{n \to \infty} \left( |\sum_{k=1}^{\infty} \alpha_k - \alpha| + \|A_n - (\alpha_k)_{k=1}^{\infty} \| \right),
\]
where $\alpha_k = \lim_{n \to \infty} a_{nk}$ for all $k \in \mathbb{N}$ and $\alpha = \lim_{n \to \infty} \sum_{k=1}^{\infty} a_{nk}$.

Corollary 5.11 ([3, Corollary 3.11]) Let $L \in B(w^p, c_0)$. Then we have
\[
\|L\|_\chi = \lim_{n \to \infty} \left( |b_{n0}| + \|(b_{nk})_{k=1}^{\infty}\|_{\mathcal{M}_p} \right)
\]
with $b_{nk} = b_{k}^{(n)} (n \in \mathbb{N}; k \in \mathbb{N}_0)$ from Theorem 5.9.

Corollary 5.12 ([3, Corollary 3.12]) Let $A \in (w^p, c_0)$. Then we have
\[
\|A\|_\chi = \|L_A\|_\chi = \lim_{n \to \infty} \|A_n\|_{\mathcal{M}_p}.
\]
Characterisations of the compact operators $L \in \mathcal{B}(X, Y)$ for $X = w_0^p, w_\infty^p$ and $Y = c_0, c$

Corollary 5.13 ([3, Corollary 3.13]) Let $L \in \mathcal{B}(X, Y)$. Then the necessary and sufficient conditions for $L \in \mathcal{C}(X, Y)$ can be read from the table

<table>
<thead>
<tr>
<th>From</th>
<th>$w_0^p$</th>
<th>$w_\infty^p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_0$</td>
<td>1.</td>
<td>2.</td>
</tr>
<tr>
<td>$c$</td>
<td>3.</td>
<td>4.</td>
</tr>
</tbody>
</table>

where

1. $\lim_{r \to \infty} (\sup_{n \geq r} \| A_n \|_{\mathcal{M}_p}) = 0$;
2. $\lim_{n \to \infty} (|b_{n0} + \| B_n \|_{\mathcal{M}_p}) = 0$;
3. $\lim_{r \to \infty} (\sup_{n \geq r} \| A_n - (\alpha_k)_{k=1}^\infty \|_{\mathcal{M}_p}) = 0$;
4. $\lim_{n \to \infty} \left( |b_{n0} - \beta + \sum_{k=1}^\infty \beta_k | + \| B_n - (\beta_k)_{k=1}^\infty \|_{\mathcal{M}_p} \right) = 0$.
Characterisations of the compact matrix operators in $(w^p, c_0)$ and $(w^p, c)$

**Corollary 5.14 ([3, Corollary 3.14])**

(a) Let $A \in (w^p, c_0)$. Then we have $L_A \in C(w^p, c_0)$ if and only if

$$\lim_{n \to \infty} \|A_n\|_{\mathcal{M}_p} = 0.$$ 

(b) Let $A \in (w^p, c)$. Then we have $L_A \in C(w^p, c)$ if and only if

$$\lim_{n \to \infty} \left( \left\| \sum_{k=1}^{\infty} \alpha_k - \alpha \right\| + \|A_n - (\alpha_k)_{k=1}^{\infty}\|_{\mathcal{M}_p} \right) = 0$$

with $\alpha_k$ ($n \in \mathbb{N}$) and $\alpha$ from Corollary 5.10.
Characterisation of the compact matrix operators in $(w_p^\infty, c_0)$ and $(w_p^\infty, c)$

**Corollary 5.15** ([3, Corollary 3.15])

(a) Let $A \in (w_p^\infty, c_0)$. Then we have $L_A \in C(w_p^\infty, c_0)$ if and only if 1. in Corollary 5.13 holds.

(b) Let $A \in (w_p^\infty, c)$. Then we have $L_A \in C(w_p^\infty, c_0)$ if and only if 3. in Corollary 5.13 holds.

**Strong $C_1$–regularity**

We call an operator $L \in B(w_p^p, c)$ **strongly $C_1$–regular**, if

$$\lim_{n \to \infty} L_n(x) = \xi \text{ for all } x \in w_p^p,$$

where $\xi$ is the strong $C_1$–limit of $x$.

A matrix $A \in (w_p^p, c)$ is said to be **strongly $C_1$–regular**, if the operator $L_A$ is strongly $C_1$–regular.
Characterisation of compact strongly $C_1$–regular operators

**Corollary 5.16** ([3, Corollary 3.16]) Let $L \in B(w^p, c)$ be strongly $C_1$–regular. Then we have $L \in C(w^p, c)$ if and only if

\[
\lim_{n \to \infty} \left( |b_{n0} - 1| + \|B_n\|_{\mathcal{M}_p} \right) = 0
\]

with $b_{nk} = b^{(n)}_k$ ($n \in \mathbb{N}; k \in \mathbb{N}_0$) from Theorem 3.16.

A strongly $C_1$–regular matrix cannot be compact

**Remark 5.17** ([3, Remark 3.17]) If $A$ is a strongly $C_1$–regular matrix then $L_A$ cannot be compact, since we have with $b_{n0} = 0$ for all $n \in \mathbb{N}_0$ and

\[
1 + \|A_n\|_{\mathcal{M}_p} \geq 1 \neq 0 \text{ for all } n,
\]

and so (5.5) in Corollary 5.16 cannot hold (Remark 3.20).
6 Graphical Representations of Neighbourhoods

We consider $\mathbb{R}^n$ for given $n \in \mathbb{N}$ as a subset of $\omega$ by identifying every point $X = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ with the real sequence $x = (x_k)_{k=1}^{\infty} \in \omega$ where $x_k = 0$ for all $k > n$, and introduce some of the metrics of the previous sections on $\mathbb{R}^n$.

Let $B_d(r, X_0) = \{X \in \mathbb{R}^n : d(X, X_0) < r\}$ and $\partial B_d(X_0)$ denote the open ball in $(\mathbb{R}^n, d)$ of radius $r > 0$ with its centre in $X_0$, and its boundary.

We use the boundaries $\partial B_d(X_0)$ for the graphical representations of neighbourhoods.
Example 6.1 We consider the metric $d(p)$ of Example 2.4 (b).

Then $\partial B_{d(p)}(r, X_0)$ is given by a parametric representation

$$\bar{x}(u_1, u_2) = (\phi_1(u_1, u_2), \phi_2(u_1, u_2), \phi_3(u_1, u_2)) + \bar{x}_0$$

for $((u_1, u_2) \in D = (-\pi/2, \pi/2) \times (0, 2\pi))$

with (Figure 6.1)

$$\phi_1(u_1, u_2) = r^{M/p_1} \cdot \text{sgn}(\cos u_2)(\cos u_1 |\cos u_2|)^{2/p_1},$$

$$\phi_2(u_1, u_2) = r^{M/p_2} \cdot \text{sgn}(\sin u_2)(\cos u_1 |\sin u_2|)^{2/p_2},$$

$$\phi_3(u_1, u_2) = \phi_3(u_1) = r^{M/p_3} \cdot \text{sgn}(\sin u_1)|\sin u_1|^{2/p_3}.$$
Representation of $\partial B_{d(p)}(X_0)$ in Example 6.1

Figure 6.1 $\partial B_{d(p)}(X_0, r)$ for

$p = (1/2, 2, 3/2); \quad p = (1/2, 4, 1/4)$
Neighbourhoods in a relative topology

Figure 6.2 Neighbourhoods in the relative topology on Enneper’s surface of the metrics $d_{(p)}$ of Figure 6.1
Neighbourhoods in a weak topology

Figure 6.3 Neighbourhoods on a sphere in the weak topology by the stereographic projection
7 Wulff’s Crystals and Potential Surfaces

Here we deal with Wulff’s construction, and the graphical representation of Wulff’s crystals and their surface energy functions as potential surfaces.

Wulff’s principle

According to Wulff’s principle ([28]), the shape of a crystal is uniquely determined by its surface energy function. A surface energy function is a real–valued function depending on a direction in space.

Let $B^n$ and $\partial B^n$ denote the unit sphere and its boundary in euclidean $\mathbb{R}^{n+1}$.
Potential surfaces

Let $F : \partial B^n \to \mathbb{R}$ be a surface energy function. Then we may consider the set

$$PM = \{ \vec{x} = F(\vec{e})\vec{e} \in \mathbb{R}^{n+1} : \vec{e} \in \partial B^n \}$$

as a natural representation of the function $F$.

If $n = 2$, then

$$\vec{e} = \vec{e}(u^1, u^2) = (\cos u^1 \cos u^2, \cos u^1 \sin u^2, \sin u^1)$$

for $(u^1, u^2) \in R = (-\pi/2, \pi/2) \times (0, 2\pi)$

and we obtain a potential surface with a parametric representation (Figure 7.1)

$$PS = \{ \vec{x} = f(u^1, u^2)(\cos u^1 \cos u^2, \cos u^1 \sin u^2, \sin u^1) : (u^1, u^2) \in R \}$$

where $f(u^1, u^2) = F(\vec{e}(u^1, u^2))$. 

77
Figure 7.1 A potential surface

\[ F(\bar{e}(u^1, u^2))\bar{e}(u^1, u^2) \]
Representations of potential surfaces and corresponding Wulff’s crystals

Figure 7.2 Potential surfaces and corresponding Wulff’s crystals
Representations of potential surfaces and corresponding Wulff’s crystals

Figure 7.3 Potential surfaces and corresponding Wulff’s crystals
Wulff’s principle

Let $\bullet$ denote the usual inner product in $\mathbb{R}^n$.

Wulff gave a geometric principle of construction for crystals.

**Theorem 7.1 (Wulff’s principle) ([8, p. 88])** For every $\vec{e} \in \partial B^n$, let $E_{\vec{e}}$ denote the hyperplane orthogonal to $\vec{e}$ and through the point $P$ with position vector $\vec{p} = F(\vec{e})\vec{e}$, and $H_{\vec{e}}$ be the half space which contains the origin and has the boundary $E_{\vec{e}} = \partial H_{\vec{e}}$. Then the crystal $C_F$ which has $F$ as its surface energy function is uniquely determined and given by

$$C_F = \bigcap_{\vec{e} \in \partial B^n} H_{\vec{e}} = \bigcap_{\vec{e} \in \partial B^n} \{ \vec{x} : \vec{x} \bullet \vec{e} \leq F(\vec{e}) \}.$$

**Remark 7.2** It is clear that if the surface energy function $F$ is continuous then $C_F$ is a closed convex subset of $\mathbb{R}^{n+1}$. 
Theorems on Wulff’s construction

**Theorem 7.3** ([8, Satz 6.1]) Let \( F : \partial B^n \to \mathbb{R}^+ \) be a continuous function. Then a point \( X \) with position vector \( \vec{x} \) is on the boundary \( \partial C_F \) of Wulff’s crystal \( C_F \) corresponding to \( F \) if and only if (left in Figure 7.4)

\[
F(\vec{e}) \geq \vec{x} \cdot \vec{e} \quad \text{for all } \vec{e} \in \partial B^n \quad \text{and} \quad F(\vec{e}_0) = \vec{x} \cdot \vec{e}_0 \quad \text{for some } \vec{e}_0 \in \partial B^n.
\]

**Theorem 7.4** ([8, Satz 6.2]) Let \( F : \partial B^n \to \mathbb{R}^+ \) be a continuous function and \( C_F : \partial B^n \to \mathbb{R}^+ \) be defined by

\[
(7.6) \quad C_F(\vec{e}) = \inf \left\{ F(\vec{u})(\vec{e} \cdot \vec{u})^{-1} : \vec{u} \in \partial B^n \quad \text{and} \quad \vec{e} \cdot \vec{u} > 0 \right\}.
\]

Then the boundary \( \partial C_F \) of Wulff’s crystal corresponding to \( F \) is given by

\[
(7.7) \quad \partial C_F = \{ \vec{x} = C_F(\vec{e})\vec{e} \in \mathbb{R}^{n+1} : \vec{e} \in \partial B^n \}.
\]
Wulff’s crystal for $n = 2$

Example 7.5 If $n = 2$, then a parametric representation for the boundary $\partial C_F$ of Wulff’s crystal corresponding to $F$ is (right in Figure 7.4)

$\bar{x}(u^1, u^2) = CF(\bar{e}'(u^1, u^2))\bar{e}'(u^1, u^2)$

for $(u^1, u^2) \in R = (-\pi/2, \pi/2) \times (0, 2\pi)$.

Wulff’s constructions according to Theorems 7.1 and 7.3

Figure 7.4 Wulff’s constructions according to Theorems 7.1 and 7.3
Potential surfaces and Wulff’s crystals constructed by Theorems 7.1 and 7.3

Figure 7.5 Wulff’s crystals constructed by Theorems 7.3 and 7.4
The special case $F = \| \cdot \|$ 

When $F$ is equal to a norm $\| \cdot \|$ in $\mathbb{R}^3$, then the boundary of the corresponding Wulff’s crystal is given by the dual norm of $\| \cdot \|$.

**Corollary 7.6** Let $\| \cdot \|$ be a norm on $\mathbb{R}^{n+1}$ and, for each $\vec{w} \in \partial B^n$, let $\phi_{\vec{w}} : \mathbb{R}^{n+1} \to \mathbb{R}$ be defined by

$$
\phi_{\vec{w}}(\vec{x}) = \vec{w} \cdot \vec{x} = \sum_{k=1}^{n+1} w_k x_k \quad (\vec{x} \in \mathbb{R}^{n+1}).
$$

Then the boundary $\partial C_{\| \cdot \|}$ of the corresponding Wulff’s crystal is given by

$$(7.9) \quad \partial C_{\| \cdot \|} = \left\{ \vec{x} = \frac{1}{\| \phi_{\vec{e}} \|^*} \cdot \vec{e} \in \mathbb{R}^{n+1} : \vec{e} \in \partial B^n \right\},$$

where $\| \phi_{\vec{e}} \|^*$ is the norm of the functional $\phi_{\vec{e}}$, hence the dual norm of $\| \cdot \|$. 

85
The dual cases $F = \| \cdot \|_1$ and $F = \| \cdot \|_\infty$

Figure 7.6 *Wulff’s crystals corresponding to the norms* $\| \cdot \|_1$ and $\| \cdot \|_\infty$
The case $F = \| \cdot \|_{w_p}$

Figure 7.7 Potential surface of the $w_p$ norm and potential surface with corresponding Wulff’s crystal
The case $F = \| \cdot \|_{\mathcal{M}_p}$

**Figure 7.8** Potential surface of the $\mathcal{M}_p$ norm and potential surface with corresponding Wulff’s crystal
References


