SOLUTIONS OF OPTIMIZATION PROBLEMS, GAMES AND TOPOLOGICAL PROPERTIES OF SPACES

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Abstract

The distinct optimization problems and topological games of the Banach - Mazur - Michael types are examined.

We present conditions under which the set of continuous perturbations of a given lower semi-continuous function attain minimum on a subset with concrete properties is "big" in a topological sense.

1 Introduction

By a space we understand a completely regular topological Hausdorff space. We use the terminology from [32, 43, 34].

Let the space of reals, $\mathbb{R}^+$ = \{x ∈ $\mathbb{R}$ : x ≥ 0\}, $\mathbb{N}$ = \{1, 2, ...\} and $I$ = [0, 1] be the subspaces of the space $\mathbb{R}$, $\mathbb{R}^\infty$ = $\mathbb{R}$ ∪ {+∞}.

On $\mathbb{R}^*$ = $\mathbb{R}$ ∪ {−∞, +∞} we consider:

- the linear order $-\infty < t < +\infty$ for each $t \in \mathbb{R}$;

- that $0 \cdot (±\infty) = (±\infty) \cdot 0 = 0$, $(-\infty) \cdot (+\infty) = (+\infty) \cdot (-\infty) = (-\infty) \cdot (\lambda) = (\lambda) \cdot (-\infty) = (+\infty) \cdot (\mu) = (\mu) \cdot (+\infty) = -\infty$ and $(-\infty) \cdot (-\infty) = (+\infty) \cdot (+\infty)$

= $(\mu) \cdot (-\infty) = (+\infty) \cdot (\lambda) = (\lambda) \cdot (+\infty) = +\infty$ for all $\lambda, \mu \in \mathbb{R}$, $\mu < 0$ and $\lambda > 0$;

- the distance $d(x, y) = \min\{|x - y|, 1\}$.

In this case the structure of the space $\mathbb{R}^*$ is similar to the structure of the closed interval $I$.

The present work contains three parts.

In the Part 1, the only for understanding the material of the second part, we introduce some fundamental facts of the metric, topological and linear spaces.

In the Part 2 we study the minimization problems on topological spaces.

In the Part 3 we determine the conditions of differentiability of the functionals of minimizing.

The Parts 2 and 3 include results from the articles [20, 21, 22, 23, 24].
2 Metrical and Topological Spaces

The concept of a topological space can be considered as an axiomatization of the notion of closeness of a point to a set and of some geometric notions.

**Definition 2.1.** A metric space is a pair \((X, \rho)\) consisting of a set \(X\) and a function \(\rho\) defined on \(\times X\) and satisfying the following conditions:

(M1) \(\rho(x, y) = 0\) if and only if \(x = y\).
(M2) \(\rho(x, y) = \rho(y, x)\) for all \(x, y \in X\).
(M3) \(\rho(x, z) \leq \rho(x, y) + \rho(y, z)\) for all \(x, y, z \in X\).

The set \(X\) is called a space, the elements of \(X\) are called points, the function \(\rho\) is called a metric on the set \(X\) and the number \(\rho(x, y)\) is called the distance between \(x\) and \(y\).

A function \(\rho\) defined on the set \(X \times X\) and satisfying the conditions (M2), (M3) and the condition

(M1') \(\rho(x, x) = 0\) for each \(x \in X\)

is called a pseudo-metric on the set \(X\). The pair \((X, \rho)\) is called a pseudo-metric space.

Any pseudo-metric is a non-negative function:

\[
2d(x, y) = d(x, y) + d(x, y) = d(x, y) + d(y, x) \geq d(x, x) = 0.
\]

If \((X, \rho)\) is a metric space, then for any point \(x \in X\) and any \(r > 0\), we define the open and closed balls:

\[
B(x, \rho, r) = \{y \in X : \rho(x, y) < r\},
\]

\[
\bar{B}(x, \rho, r) = \{y \in X : \rho(x, y) \leq r\}.
\]

Suppose that \((X, \rho)\) is a metric space. Then \(X\) carries a natural topology \(T(\rho)\) constructed as follows. We say that a set \(U \subseteq X\) is \(\rho\)-open or simple open, if it has the property: for every \(x \in U\) there exists some \(r = r(x, U) > 0\) such that \(B(x, \rho, r) \subseteq U\). The family \(T(\rho)\) is the collection of all \(\rho\)-open subsets of \(X\). One can prove that the collection \(T(\rho)\) has the following properties:

(T1). \(X \in T(\rho)\).
(T2). \(\emptyset \in T(\rho)\).
(T3). If \(\{U_\alpha : \alpha \in A\} \subseteq T(\rho)\) is a family of open sets, then the union \(\bigcup\{U_\alpha : \alpha \in A\} \in T(\rho)\) is again open.
(T4). If \(U_1\) and \(U_2\) are open, then the intersection \(U_1 \cap U_2 \in T(\rho)\) is again open.
(T5). The balls \(B(x, \rho, r)\) are open sets.

The topology \(T(\rho)\) thus constructed is called the metric topology.

A topology on the set \(X\) is a family \(T\) of subsets of \(X\) with the properties \(T1 - T4\). The sets from \(T\) are called open in the topological space \(X\). A set \(F \subseteq X\) is called closed in the space \(X\) if its complement \(X \setminus F\) is an open set.

The family \(cT\) of all closed sets has the following properties:

(C1). \(X \in cT\).
(C2). \(\emptyset \in cT\).
(C3). If \( \{ F_\alpha : \alpha \in A \} \subset cT \) is a family of closed sets, then the intersection 
\( \bigcap \{ F_\alpha : \alpha \in A \} \in cT \) is again closed.

(C4). If \( F_1 \) and \( F_2 \) are closed sets, then the union \( F_1 \cup F_2 \in cT \) is again closed.

Sets which are simultaneously open and closed in the space \( X \) are called open-and-closed or closed-and-open sets.

Countably unions of closed sets are called \( F_\sigma \)-sets. Countably intersections of open sets are called \( G_\delta \)-sets. The complement of a \( G_\delta \)-set is an \( F_\sigma \)-set and vice versa.

The notion of continuity is the fundamental notion in topology and can be defined for a mapping between topological spaces.

**Definition 2.2.** The mapping \( f : X \longrightarrow Y \) between two topological spaces is continuous if for every open (in the topology on \( Y \)) set \( U \subset Y \) the inverse image \( f^{-1}(U) \) is open in the topology on \( X \).

If both the mapping \( f : X \longrightarrow Y \) and its inverse \( f^{-1} : Y \longrightarrow X \) are continuous, then the mapping \( f \) is called a homeomorphism. More precisely we have: The bijection (one-to-one and onto mapping) \( f : X \longrightarrow Y \) between two topological spaces is called homeomorphism if it is true that \( f^{-1}(U) \) is open in \( X \) if and only if \( U \) is open in \( Y \).

Any homeomorphism \( f : X \longrightarrow X \) is called a continuous deformation of a space \( X \), i.e., deformation where it is not allowed to cut objects and glue them together.

Two topological spaces \( X \) and \( Y \) are called homeomorphic if there exists a homeomorphism between \( X \) and \( Y \).

Topology is a major area of mathematics. In the topology we study the properties of objects which are not sensitive to continuous deformations. These properties are called topological properties. In general, a property of topological spaces is a topological property if it is preserved under homeomorphisms.

A topological space \((X, T)\) is called metrizable if there exists a metric \( d \) on \( X \) such that \( T = T(d) \). And here is one of the important problems in topology: **which topological spaces are metrizable?**

Another important question in Topology is: **how do we decide if two spaces are homeomorphic or not?**

Let \((X, T)\) be a topological space and \( Z \subset X \). Then \( T|Z = \{ U \cap Z : U \in T \} \) is a topology on the set \( Z \) and the pair \((Z, T|Z)\) is called a subspace of the space \((X, T)\).

A continuous mapping \( f : X \longrightarrow Y \) (between two topological spaces) is called an embedding if \( f \) is injective and, as a function from \( X \) to its image \( f(X) \), it is a homeomorphism of \( X \) onto the subspace \( f(X) \) of the space \( Y \). There are injective continuous maps that are not embeddings.

Again, one of the important questions in Topology is: **understand when a space \( X \) can be embedded in another given space \( Y \)?**

**Example 2.3.** On the space of real numbers \( \mathbb{R} \) consider the Euclidean topology generated by the Euclidean metric \( d(x, y) = |x - y| \). A continuous mapping
$f : X \rightarrow \mathbb{R}$ is called a continuous functions on a space $X$. By $C(X)$ we denote the space of all continuous bounded functions on a space $X$ with the sup-norm $\|f\| = \text{sup} \{|f(x) : x \in X\}$ and the metric $d_s(f, g) = \|f - g\|$.

Example 2.4. Let $n \geq 1$ and $\mathbb{E}^n$ be the set of all points of the form $x = (x_1, x_2, ..., x_n)$, where $x_1, x_2, ..., x_n \in \mathbb{R}$. On $\mathbb{E}^n$ consider the following metrics:

- the Euclidean metric $d((x_1, x_2, ..., x_n), (y_1, y_2, ..., y_n)) = (\Sigma \{|x_i - y_i|^2 : i \leq n\})^{1/2}$;
- the taxi-cab metric $d_c((x_1, x_2, ..., x_n), (y_1, y_2, ..., y_n)) = \Sigma \{|x_i - y_i| : i \leq n\}$;
- the supremum or maximum metric $d_s((x_1, x_2, ..., x_n), (y_1, y_2, ..., y_n)) = \max\{|x_i - y_i| : i \leq n\}$;

We have $T(d) = T(d_c) = T(d_s)$. But the metrics $d, d_c$ and $d_s$ are distinct geometrical properties.

Remark. Let $(X, \rho)$ be a metric space, $F$ and $\Phi$ be closed subsets of the space $X$, and $F \cap \Phi = \emptyset$. Then:

1. The number $\rho(x, F) = \inf \{\rho(x, y) : y \in F\}$ is called the distance between the point $x$ and the set $F$.
2. The function $h_F(x) = \rho(x, F)$ is continuous on $X$ and $h_F(x) = 0$ if and only if $x \in F$.
3. The function $g(x) = h_F(x)/(h_F(x) + H_\Phi(x))$ has the following properties:
   - the function $g$ is continuous;
   - $0 \leq g(x) \leq 1$ for each $x \in X$;
   - $g(x) = 0$ if and only if $x \in F$;
   - $g(x) = 1$ if and only if $x \in \Phi$.
4. Any finite subset of the space $X$ is closed.

The topological spaces with the properties 3 and 4 are called normal spaces.

Definition 2.5. A topological space $X$ is called a Tychonoff or a completely regular space if it has the following properties:

1. Any finite subset of the space $X$ is closed.
2. If $F$ is a closed subset of the space $X$ and $x_0 \in X \setminus F$, then there exists a continuous function $g : X \rightarrow \mathbb{R}$ with the following properties:
   - $0 \leq g(x) \leq 1$ for each $x \in X$;
   - $g(x_0) = 0$;
   - $g(x) = 1$ for each $x \in F$.

Any normal space is a Tychonoff space. Thus any metrizable space is a normal space and a Tychonoff space.

3 Closure operations

Let $((X, T)$ be a topological space.

The closure of a set $A$ in the space $X$ is the smallest closed set $\text{cl}_X A$ containing $A$, i.e. $\text{cl}_X A = \cap \{F : A \subset F, X \setminus F \in T\}$. 

4
If the set $U$ is open and $x \in U$, then $U$ is called a neighbourhood of the point $x$ in the space $X$.

A family $\mathcal{B}$ of subsets of a space $X$ is called:
- a base of the space $X$ at a point $b \in X$ if all sets $W \in \mathcal{B}$ are open in $X$ and for each neighbourhood $U$ of the point $b$ in the space $X$ there exists $V \in \mathcal{B}$ such that $b \in V \subset U$;
- a base of the space $X$ if $\mathcal{B}$ is a base of the space $X$ at any point $x \in X$.

A space $X$ is a first-countable space if $X$ has a countable base at each point. A space $X$ is a second-countable space if $X$ has a countable base.

**Proposition 3.1.** Let $A$ be a subset of the space $Z$ and $x \in X$. Then the following assertions are equivalent:
1. $x \in cl_X A$.
2. $U \cap A \neq \emptyset$ for each neighbourhood $U$ of the point $x$ in $X$.

Let $A \subseteq X$. The set $Fr_X A = cl_X A \cap cl_X (X \setminus A)$ is called the boundary of the set $A$ in the space $X$. The set $Int_X A = A \setminus cl_X (X \setminus A)$ is called the interior of the set $A$ in the space $X$. We have:
- $F R_X A = cl_X A \setminus Int_X A$;
- $A = Int_X A$ if and only if the set $A$ is open in $X$.

A set $A$ is dense in $X$ if $cl_X A = X$. A space $X$ is separable if $X$ has a dense countable subset. Any second-countable space is separable.

**Proposition 3.2.** The closure operator has the following properties:
1. $cl_X \emptyset = \emptyset$.
2. $A \subseteq cl_X A$.
3. $cl_X (A \cup B) = cl_X A \cup cl_X B$.
4. $cl_X (cl_X A) = cl_X A$.

**Proposition 3.3.** Suppose that $f : X \longrightarrow Y$ is a mapping between topological spaces. Then the following assertions are equivalent:
1. $f$ is continuous.
2. $F(cl_X A) \subseteq cl_Y f(A)$ for any subset $A \subseteq X$.
3. if the $F$ is closed in $Y$, then the set $f^{-1}(F)$ is closed in $X$.
4. if the $U$ is open in $Y$, then the set $f^{-1}(U)$ is open in $X$.
5. For every point $x \in X$ and each neighbourhood $V$ of the point $y = f(x)$ in $Y$ there exists a neighbourhood $U$ of the point $x$ in $X$ such that $f(U) \subset V$.

Let $X$ be a topological space. A point $b \in X$ is called a limit point of the sequence $\{x_n \in X : n \in \mathbb{N}\}$ and we say that the sequence $\{x_n : n \in \mathbb{N}\}$ is convergent to the point $b$ if for each neighbourhood $U$ of the point $b$ in $X$ there exists a number $m \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq m$. In this case we put $b = \lim_{n \to \infty} x_n$.

There exists topological spaces without non-trivial convergent sequences. A sequence $\{x_n \in X : n \in \mathbb{N}\}$ is called trivial or almost-constant if there exists $m \in \mathbb{N}$ such that $x_n = x_m$ for all $n \geq m$.

A space $X$ is called sequential if for each non-closed subset $A$ of $X$ there exist a sequence $\{x_n \in A : n \in \mathbb{N}\}$ and a point $b \in X \setminus A$ such that $b = \lim_{n \to \infty} x_n$. 

5
A space \( X \) is called a Frechét-Urysohn space if for each subset \( A \) of \( X \) and any point \( b \in \text{cl}_{X} A \) there exists a sequence \( \{x_{n} \in A : n \in \mathbb{N} \} \) such that \( b = \lim_{n \to \infty} x_{n} \).

Any metric space is a first-countable space, any first-countable space is a Frechét-Urysohn space, any Frechét-Urysohn space is a sequential space. All inverse implications are not true.

**Remark.** Let \( \text{exp}(X) \) be the family of all subsets of the set \( X \). In the solving of many mathematical problems aper operators \( c : \text{exp}(X) \to \text{exp}(X) \) with some properties of the following type:

1. \((C1)\) \( c(\emptyset) = \emptyset \).
2. \((C2)\) If \( A \subset B \), then \( c(A) \subset c(B) \).
3. \((C3)\) \( c(A \cup B) = c(A) \cup c(B) \).
4. \((C4)\) \( c(c(A)) = c(A) \).

Obviously, \((C3) \to (C2)\). Operator \( c \) with the properties \((C1) - (C4)\) is called a Kuratowski closure operator [43]. Operator \( c \) with the properties \((C1) - (C3)\) is called a Čech closure operator.

**Remark.** Let \( X \) be a space. The sequential closure \( s-\text{cl}_{X} A \) is the collection of all points \( b \in X \) such that \( b = \lim_{n \to \infty} x_{n} \) for some sequence \( \{x_{n} \in A : n \in \mathbb{N} \} \). Obviously, \( s-\text{cl}_{X} A \subseteq \text{cl}_{X} A \) for each subset \( A \subset X \). The operation \( s-\text{cl}_{X} \) is a Čech closure operator. If \( X \) is a Frechét-Urysohn space, then the operation \( s-\text{cl}_{X} \) is a Kuratowski closure operator.

### 4 Topologically complete spaces

Let \((X, \rho)\) be a metric space.

We say that \( \{x_{n} \in X : n \in \mathbb{N} \} \) is a Cauchy sequence in \( X \) if for every \( \epsilon > 0 \) there exists a natural number \( m \in \mathbb{N} \) such that \( \rho(x_{n}, x_{k}) < \epsilon \) for all \( n, k \geq m \).

Any convergent sequence is a Cauchy sequence in \( X \).

A metric space \((X, \rho)\) is called a complete metric space if every Cauchy sequence in \( X \) is convergent to some point of \( X \).

A topological space is called completely metrizable if there exists a complete metric on \( X \).

Obviously, a closed subspace of a completely metrizable space is completely metrizable.

A \( G_{\delta} \)-subspace of a completely metrizable space is completely metrizable (P. S. Alexandroff - F. Hausdorff).

If a completely metrizable space \( X \) is a dense subspace of a space \( Y \), then \( X \) is a \( G_{\delta} \)-subspace of the space \( Y \) (M. Lavrentieff).

**Example 4.1.** Let \( X \) be any set. The normed discrete metric on the \( X \) is given by: \( d(x, y) = 0 \) if \( x = y \) and \( d(x, y) = 1 \) otherwise. The discrete metric space is a complete metric space. On the infinite discrete space there exists a non-complete metric too.
Example 4.2. A metric $\rho$ on a set $X$ is called discrete if there exists a number $c > 0$ such $d(x, y) \geq c$ for all distinct points $x, y \in X$. The discrete metric space is a complete metric space. On the infinite discrete space there exists a non-complete metric too.

Example 4.3. A metric space $(X, \rho)$ is called topologically discrete if $T(\rho) = exp(X)$. If $(X, \rho)$ is a topologically discrete space, then $X$ is completely metrizable. There exists a complete topologically discrete metric space $(Z, \rho)$ which is not discrete. For example, the set $Z = \{(n, 0) : n \in \mathbb{N}\} \cup \{(n, n^{-1}) : n \in \mathbb{N}\}$ as a metric subspace of the Euclidean plain $\mathbb{E}^2$ is topologically discrete metric space with a non-discrete metric.

Example 4.4. In some cases it is necessity to examine the metrics $d$ with infinite distance $d(x, y) = +\infty$ between points. Let $X$ be a topological space. Denote by $C^*(X)$ the space of all continuous functions on $X$. Let $||g|| = sup\{|g(x)| : x \in X\}$ for each $g \in C^*(X)$. In this case $d_s(f, g) = ||f - g||$ is a metric on $C^*(X)$ which admits and infinite distance between some points. The metric space $(C^*(X), d_s)$ is complete.

Example 4.5. Let $X$ be a topological space. Denote by $C(X)$ the space of all bounded continuous functions on $X$. Let $||g|| = sup\{|g(x)| : x \in X\}$ for each $g \in C(X)$. In this case $d_s(f, g) = ||f - g||$ is a metric on $C^*(X)$ and $d_s(f, g) < \infty$ for all $f, g \in C(X)$. The metric space $(C(X), d_s)$ is complete. Moreover, $(C(X), d_s)$ is a closed subspace of the metric space $(C^*(X), d_s)$.

Example 4.6. The space $\mathbb{Q}$ of rational numbers, with the standard metric given by the absolute value of the difference, is not a complete metric space. complete metrizable. The space $\mathbb{I} = \mathbb{R} \setminus \mathbb{Q}$ of irrational numbers, with the standard metric given by the absolute value of the difference $d(x, y) = |x - y|$ is not a complete metric space. The topological space $\mathbb{Q}$ is not completely metrizable and The topological space $\mathbb{I}$ is completely metrizable.

Example 4.7. The open interval $I^* = (0, 1)$, again with the absolute value metric $d(x, y) = |x - y|$ is not complete either. The space $I^*$ is completely metrizable. However the closed interval $I = [0, 1]$ is complete, the sequence $\{n^{-1} : n \in \mathbb{N}\}$ does not have a limit in the interval $I^*$ and the limit is zero. The space $\mathbb{R}$ of real numbers and the space $\mathbb{C}$ of complex numbers (with the metric given by the absolute value) are complete, and so is Euclidean space $\mathbb{E}^n$, with the usual distance metric.

Every metric space $X$ is a dense subspace of a complete metric space $Y$ which is called a metric completion of $X$. The metric completion is unique (F. Hausdorff).

A topological space $X$ is called a space with the Baire property if for any sequence $\{U_n : n \in \mathbb{N}\}$ of dense open subsets of $X$ the intersection $Y = \{U_n : n \in \mathbb{N}\}$ is dense in $X$.

The Baire category theorem says that every complete metric space is a space with the Baire property (R. Baire for real line $\mathbb{R}$, F. Hausdorff for complete metric spaces, E. Čech for Čech-complete spaces see [32]).
A subset $A$ of a space $X$ is nowhere dense subset if $X \setminus cl_X A$ is dense in $X$. Hence, in a Baire space the union of countably many nowhere dense subsets of the space has empty interior.

The Baire category theorem is an efficient and often used method of establishing the existence of some objects (it is called the category method).

5 Compactness and classes of spaces

A point $x$ in a topological space $X$ is called an accumulation point of a set $A$ if $x \in cl_X (A \setminus \{x\})$.

A space $X$ is called a countably compact space if any infinite subset of $X$ has an accumulation point.

A subset $F$ of a space $X$ is called a compact subset of $X$ if for any family $\gamma$ of open subsets of $X$ such that $F \subset \cup \gamma$ there exist a finite number of sets $U_1, U_2, \ldots, U_n \in \gamma$ for which $F \subset \cup \{U_i : i \leq n\}$. A such family $\gamma$ of open subsets of $X$ is called an open cover of the set $F$ in $X$. Thus a set $F$ is compact if and only if any open cover of $F$ in $X$ contains a finite subcover.

A space $X$ is a compact space if $X$ is a compact subset of the space $X$.

Any compact space is a normal space. A closed subset of a compact space is a compact subset. Any compact subset is a closed subset.

A metrizable space is compact if and only if it is countably compact.

A sequence $\{U_n : n \in \mathbb{N}\}$ of subsets of a space $X$ is called a convergent sequence on $X$ (see [4, 3, 7]) if:

(i) any sequence $\{x_n \in U_n \cap X : n \in \mathbb{N}\}$ has an accumulation point in $X$;

(ii) $\cap \{U_n : n \in \mathbb{N}\}$ is a compact set.

A subset $F$ of the space $X$ is called a set with the property $k$ in $X$ if the subspace $F$ is compact.

A point $x \in X$ is called a k-point or a point of countable type if there exists a compact subset $F$ and a convergent sequence $\{U_n : n \in \mathbb{N}\}$ of open subsets of $X$ such that $x \in F = \cap \{U_n : n \in \mathbb{N}\}$ [2]. In this case we say that the set $F$ is of the countable character in $X$.

A space $X$ is called a space of pointwise countable type if each point $x \in X$ is a point of countable type [1, 2].

Any topological space $X$ is a dense subspace of some compact space $Z$. A such space $Z$ is called a compactification of the space $X$.

M. H. Stone and E. Čech has proved that for any topological space $X$ there exists a such compactification $\beta X$ such that: for any continuous mapping $f : X \rightarrow Y$ of the space $X$ into a space $Y$ there exists a continuous mapping $\beta f : \beta X \rightarrow \beta Y$ such that $f = \beta f|X$, i. e. $f(x) = \beta f(x)$ for each $x \in X$.

The space $\beta X$ is called the Stone-Čech compactification of the space $X$.

A space $X$ is called a Čech-complete space if $X$ is a $G_\delta$-subset of some compact space.
6 Mappings and classes of spaces

Let $Q$ be a property of subsets of spaces.

Suppose that $f : X \to Y$ is a mapping between topological spaces. A mapping $f$ is called:
- an open mapping if the set $f(U)$ is open in $Y$ for each open subset $U$ of the space $X$;
- a closed mapping if the set $f(L)$ is closed in $Y$ for each closed subset $L$ of the space $X$;
- a compact mapping or a mapping with compact fibers if the set $f^{-1}(y)$ is compact for each point $y \in Y$;
- a uniformly complete mapping if there exists a Čech-complete space $Z$ such that $X$ is a subspace of $Z$ and the set $f^{-1}(y)$ is closed in $Z$ for each point $y \in Y$;
- a perfect mapping if $f(X) = Y$ and $f$ is a continuous closed mapping with compact fibers;
- a $Q$-perfect mapping if $f$ is a perfect mapping and the set $f^{-1}(y)$ has the property $Q$ for each point $y \in Y$.

Denote by $θ$ the property to be a singleton space. Any compact mapping is uniformly complete.

A space $X$ is called a (complete) $p(Q)$-space if there exist a (completely) metrizable space $Y$ and a $Q$-perfect mapping $f : X \to Y$ (see [1, 2, 16]). Any $p(Q)$-space is a paracompact space.

A space $X$ is called a complete $A(Q)$-space if there exist a complete $p(Q)$-space $Z$ and an open continuous mapping $f : Z \to X$ of the space $Z$ onto $X$ (see [16]).

A space $X$ is called an $A(\emptyset)$-space if there exist a $p(Q)$-space $Z$ and an open continuous uniformly complete mapping $f : Z \to X$ of the space $Z$ onto $X$ (see [16]).

A space $X$ is a (complete) $A(\emptyset)$-space if and only if the space $X$ has a (complete) base of countable order (see [6]).

A complete $A(k)$-space is called a sieve-complete space.

These classes of spaces were studied in [3, 4, 5, 6, 7, 15, 16, 38, 50, 60, 48, 49].

Remark. We mention the next properties of spaces:
1. If a space $X$ is sieve-complete, then $X$ contains a dense Čech-complete paracompact subspace.
2. If a space $X$ is $\emptyset$-complete, then $X$ contains a dense complete metrizable subspace.

Remark. Let $Q$ be a property hereditary relatively to closed non-empty subspaces and any sequence \( \{ U_n : n \in \mathbb{N} \} \) with the property $Q$ is a convergent sequence. We mention the next properties of (complete) $A(Q)$-spaces:
1. A non-empty closed subspace of an (a complete) $A(Q)$-space is an (a complete) $A(\emptyset)$-space.
2. A non-empty $G_δ$-subspace of an (a complete) $A(Q)$-space is an (a complete) $A(Q)$-space.
3. A locally (complete) $A(Q)$-space is an (a complete) $A(Q)$-space.
Remark. Let $X$ be a subspace of the space $Z$, $\gamma = \{\gamma_n = \{U_n : \alpha \in A_n\} : n \in \mathbb{N}\}$ be a sequence of open families of $Z$, and let $\pi = \{\pi_n : A_{n+1} \to A_n : n \in \omega\}$ be a sequence of mappings. A sequence $\alpha = \{\alpha_n : n \in \mathbb{N}\}$ is called a $c$-sequence if $\alpha_n \in A_n$ and $\pi_n(\alpha_{n+1}) = \alpha_n$ for every $n \in \mathbb{N}$. The $c$-sequence $\alpha = \{\alpha_n : n \in \mathbb{N}\}$ is called a $mc$-sequence if $H(\alpha) = \cap\{X \cap U_{\alpha_n} : n \in \mathbb{N}\}$ is a non-empty subset of the space $X$. Consider the following conditions:

1. (SC1) The set $X \subseteq \cup\{U_\beta : \beta \in A_n\}$ for each $n \in \mathbb{N}$.
2. (SC2) $\cup\{U_\beta : \beta \in \pi_n^{-1}(\alpha)\} \subseteq \cup\{d_\beta U_\beta : \beta \in \pi_n^{-1}(\alpha)\}$ for all $n \in \mathbb{N}$ and $\alpha \in A_n$.
3. (SC3) For all $n \in \mathbb{N}$, $\alpha \in A_n$ and $\beta \in q^{-1}(\alpha)$ the sets $Z \setminus V_\alpha$ and $V_\beta$ form a pair of completely separated subsets of the space $Z$.

(C1) For any $mc$-sequence $\alpha = \{\alpha_n \in A_n : n \in \mathbb{N}\}$, the sequence $\{U_{\alpha_n} : n \in \mathbb{N}\}$ is a convergent sequence with the property $\Omega$.

(C2) Any $c$-sequence $\alpha = \{\alpha_n \in A_n : n \in \mathbb{N}\}$ is an $mc$-sequence. The sequences $\gamma$ and $\pi$ are called an $A$-sieve of the space $X$ in $Z$ if they are Properties (SC1), (SC2) and (SC3).

A space $X$ is a complete $A(\Omega)$-space if on $X$ there exists an $A$-sieve with the Properties (C1) and (C2) (see [16]). A space $X$ is an $A(\Omega)$-space if on $X$ there exists an $A$-sieve with the Property (C1) (see [16]).

7 Linear spaces

Let $L$ be a nonempty set of elements on which the operations of addition and scalar multiplication are defined. That is, for any elements $x, y \in L$, the operation of addition, i.e., $x + y$, is defined, and $x + y \in L$. Similarly, for any real number $t \in \mathbb{R}$, the operation of scalar multiplication, i.e., $tx$, is defined, and $tx \in L$. The set $L$ with these two operations is called a real linear space or a vector space if the following axioms are satisfied:

1. $x + y = y + x$ for all $x, y \in L$.
2. $(x + y) + z = x + (y + z)$ for all $x, y, z \in L$.
3. There is a unique element in $L$, denoted by $0_L$ or $0$, such that $x + 0_L = x$ for all $x \in L$.
4. For each $x \in L$ there is a unique element, denoted by $-x$ in $L$ such that $x + (-x) = 0$.
5. $t(x + y) = tx + ty$ for all $x, y \in L$ and $t \in \mathbb{R}$.
6. $(u + v)x = ux + vx$ for all $x \in L$ and $u, v \in \mathbb{R}$.
7. $u(vx) = (uv)x$ for all $x \in L$ and $u, v \in \mathbb{R}$.
8. $1x = x$ for all $x \in L$.

As we will soon see, there are many more applications of linear spaces than sets of vectors in $n$-dimensional Euclidean space. The elements of a linear space are often called vectors.

The important properties about linear spaces, as far as we are concerned, are that:
- if $x, y \in L$ then $x + y \in L$;
- if \( x \in L \) and \( t \in \mathbb{R} \), then \( tx \in L \).

Let \( L \) be a linear space and let \( E \) be a non-empty subset of \( L \). \( E \) is a linear subspace of \( L \) if:

(i) if \( x, y \in E \) then \( x + y \in E \);
(ii) \( 0 \in E \);
(iii) if \( x \in E \) and \( t \in \mathbb{R} \), then \( tx \in E \).

Note that the smallest subspace of any linear space is the set consisting only of the zero element. A subspace of a linear space is a linear space.

A mapping \( g : L \rightarrow E \) of a linear space \( L \) into a linear space \( E \) is called a linear mapping, if \( g(x + y) = g(x) + g(y) \) and \( g(tx) = tg(x) \) for all \( x, y \in L \) and \( t \in \mathbb{R} \). In this case \( g(L) \) is a linear subspace of \( E \) and \( \text{Ker}(g) = \{ x \in L : g(x) = 0 \} \) is a linear subspace of \( L \).

A linear function \( f : L \rightarrow \mathbb{R} \) is called a linear functional. Let \( \text{Fun}(L) \) be the collection of all linear functionals.

A linear space \( B \) with a real-valued function \( \| \cdot \| : B \rightarrow \mathbb{R} \) so that

(N1) \( \| x + y \| \leq \| x \| + \| y \| \) (triangle inequality);
(N2) \( \| tx \| = |t| \| x \| \);
(N3) \( \| x \| = 0 \) if and only if \( x = 0 \) (positivity)
is a normed linear space, or simply normed space.

Because of the triangle inequality, the function \( d(x; y) = \| x - y \| = \| y - x \| \) is a metric. When the space \( B \) is complete with respect to this metric, \( B \) is called a Banach space.

There is a natural norm on the collection of all continuous linear maps \( g : L \rightarrow E \) from a normed space \( L \) into a normed space \( E \): \( \| f \| = \sup\{ \| g(x) \| : x \in L, \| x \| \leq 1 \} \). Let \( \text{Hom}(L, E) \) denote the collection of all continuous linear maps from the normed space \( L \) into the normed space \( E \). Then \( \text{Hom}(L, E) \) is a normed space. If \( E \) is a Banach space, then \( \text{Hom}(L, E) \) is a Banach space too.

We put \( \text{Fun}_c(L) = \text{Hom}(L, \mathbb{R}) \). Then \( \text{Fun}_c(L) \subset \text{Fun}(L) \).

Let \( X \) be a topological space. The space \( C(X), \| \cdot \| \) is a Banach space. If on the space \( X \) there exists some non-bounded continuous function, then the space \( C^*(X), \| \cdot \| \) is not a normed space.

A set \( A \subset L \) is called convex if \( tx + (1 - t)x \in A \) for all \( x, y \in A \) and \( t \in I \).

The function \( f \) is called convex (respectively, concave), if \( f(tx + (1 - t)x) \leq tf(x) = (1 - t)f(y) \) (respectively, \( f(tx + (1 - t)x) \geq tf(x) = (1 - t)f(y) \) provided \( x, y \in X \) and \( t \in I \).

A function \( f : L \rightarrow \mathbb{R} \) is affine if there exist \( g \in \text{Fun}(L) \) and a constant \( c \in \mathbb{R} \) such that \( f(x) = g(x) + c \) for all \( x \in L \).
8 Optimization Problems

Let $X$ be a subset of a set $Z$.

For any function $f : X \to \mathbb{R}^\infty$ and $Y \subseteq X$ we put $m_Y(f) = \inf\{f(x) : x \in Y\}$, $M_Y(f) = \{x \in Y : f(x) = m_Y(f)\}$ and $\text{dom}(f) = \{x \in X : f(x) < +\infty\}$.

The function $f$ is called proper if her domain $\text{dom}(f)$ is non-empty.

Fix a proper function $\psi : X \to \mathbb{R}^\infty$ and a subset $E \subseteq X$.

A pair $(X, \psi)$ is called a minimization problem, or an optimization problem, or a programming problem.

Let $E \subseteq X$. A sequence $\{x_n \in E : n \in \mathbb{N}\}$ is called a $E$-minimizing sequence of the function $\psi$ if $\lim_{n \to \infty} \psi(x_n) = \inf f(\psi)$. An $X$-minimizing sequence is called a minimizing sequence of the function $\psi$.

**Definition 8.1.** An optimization problem $(X, f, g_1, g_2, \ldots, g_m)$ in standard form contain:

- a minimization problem $(X, f)$;
- subject to $g_i(x) \leq 0, i \in \{1, 2, \ldots, m\}$, where the functions $\{g_i : X \to \mathbb{R} : i \in \{1, 2, \ldots, m\}\}$ are given and are called the constraints;
- the set $D = \{x \in \text{dom}(f) : g_i(x) \leq 0, i \in \{1, 2, \ldots, m\}\}$ is considered non-empty and it is called the domain of the given optimization problem.

An optimization problem is unconstrained if $m = 0$. Thus any minimization problem is an unconstrained optimization problem.

**Definition 8.2.** Let $(X, f, g_1, g_2, \ldots, g_m)$ be an optimization problem. Then:

- $m_X(f) = \inf\{f(x) : x \in D\}$ is called the optimal value;
- $M_X(f) = \{x \in D : f(x) = m_X(f)\}$ is the set of optimal solutions of the problem.

**Proposition 8.3.** Any optimization problem $(X, f, g_1, g_2, \ldots, g_m)$ is equivalent to some minimization problem $(X, \psi)$.

**Proof.** We put $\psi(x) = f(x)$ if $x \in D$ and $\psi(x) = +\infty$ if $x \notin D$. The optimization problems $(X, f, g_1, g_2, \ldots, g_m)$ and $(X, \psi)$ are the same optimal values, optimal solutions and minimizing sequence.

Thus we can consider only the optimizations problems of the form $(X, \psi)$.

Assume that $\mathcal{P}$ is a property of subsets of the set $X$.

The minimization problem $(X, \psi)$ has the property $\mathcal{P}$ and we denote $(X, \psi) \in \mathcal{P}$ if the set $M_X(\psi)$ has the property $\mathcal{P}$ and the sequence $\{m(x, 2^{-n}) = \{x \in X : \psi(x) \leq m_X(\psi) + 2^{-n}\}$ has the property $\mathcal{P}$.

For some cases the set $M_X(\psi)$ is empty or $(X, \psi) \notin \mathcal{P}$. In the cases of that types we change the problem $(X, \psi)$ with other problem $(X, \psi + g)$, where $g \in B$ and $B$ is a given class of bounded functions on the set $Z$. The problem $(X, \psi + g)$ is called the perturbation problem of the problem $(X, \psi)$.
Assume that the objects $X$, $Z$, $\psi$, $B$ and $\mathcal{P}$ are given. On $B$ we consider the sup-norm convergence.

We put $S_{(\mathcal{P},B)}(\psi) = \{ g \in C(Z) : M_X(f + g) \in \mathcal{P} \}$ and $S_{(\mathcal{P},B)}(f) = \{ g \in C(Z) : M_X(f + g) \neq \emptyset \}$.

We consider that $f + g$ is a function on $X$ and $(f + g)(x) = f(x) + g(x)$ for each $x \in X$.

The following questions are natural:

Q1. Under which conditions the set $S_{(\mathcal{P},B)}(\psi) = \{ g \in B : M_X(\psi + g) \in \mathcal{P} \}$ is non-empty?

Q2. Under which conditions the set $S_{(\mathcal{P},B)}(f)$ has the given properties in $B$?

To solving effective that problems we must to have some concrete spacial and algebrical structures on $X$. In dependence of the structures on $X$ and of the properties of functions $f, g_1, g_2, ..., g_m$ there exist various classes of optimal problems.

Let $X$ be a subspace of a topological space $Z$. Denote by $C(Z|X)$ the space of all continuous bounded functions on the space $Z$ with the topology generated by the sup-norm $\| g \| = \sup \{ |g(z)| : z \in Z \}$ and the respective distance $d_s(f,g) = \| f - g \|$. A function $f : X \to \mathbb{R}^\infty$ is called lower semi-continuous (respectively, upper semi-continuous) if and only if the set $\{ x \in X : f(x) > t \}$ (respectively, $\{ x \in X : f(x) < t \}$) is an open (a closed) set of $X$ for every $t \in \mathbb{R}$.

If $f : X \to \mathbb{R}^\infty$ is a lower semi-continuous function on $X$ and $g \in C(Z|X)$, then $f + g$ is a a lower semi-continuous function on $X$ and $(f + g)(x) = f(x) + g(x)$ for each $x \in X$. In this case the set $M_X(f + g)$ is closed in $X$.

An optimization problem $(X, f, g_1, g_2, ..., g_m)$ is called:

- a convex optimization problem if $Z$ is a normed space, $X$ is a convex subset of $Z$ and $f, g_1, g_2, ..., g_m$ are convex functions;
- linear programming, a type of convex programming, studies the case in which the objective function $f$ is linear and the set of constraints is specified using only linear equalities and inequalities. Such a set is called a polyhedron or a polytop if it is bounded;
- non-linear programming studies the general case in which the objective function or the constraints or both contain non-linear parts;
- infinite-dimensional optimization studies the case when the set of feasible solutions is a subset of an infinite-dimensional space.

There are well known and other classes of optimization problems: Second order cone programming; Semidefinite programming; Conic programming; Geometric programming; Integer programming; Quadratic programming; Fractional programming; Stochastic programming; Robust programming; Combinatorial optimization; Stochastic optimization; Mathematical programming with equilibrium constraints.

Optimal control theory is a generalization of the calculus of variations. Dynamic programming studies the case in which the optimization strategy is based on splitting the problem into smaller subproblems.
Example 8.4. Let \( \psi \) be a proper lower semi-continuous function on a space \( X \). A minimization problem \((X, \psi)\) has the property \( k \) if:

- \((k1)\) the set \( M_X(\psi) \) is compact;
- \((k2)\) each minimizing sequence \( \{x_n \in X : n \in \mathbb{N}\} \) of the function \( \psi \) has an accumulation point in \( X \).

In this case \( M_X(\psi) = cl_X M_X(\psi) \subseteq dom(\psi) \), the minimization problem \((X, \psi)\) is called almost-well-posed and the function \( \psi \) attains the strong minimum.

Example 8.5. Let \( \psi \) be a proper lower semi-continuous function on a space \( X \). A minimization problem \((X, \psi)\) has the property \( \theta \) if:

- \((\theta1)\) the set \( M_X(\psi) \) is a singleton;
- \((\theta2)\) each minimizing sequence \( \{x_n \in X : n \in \mathbb{N}\} \) of the function \( \psi \) has an accumulation point in \( X \).

In this case \( M_X(\psi) \subseteq dom(\psi) \), the minimization problem \((X, \psi)\) is Tychonoff well-posed and the function \( \psi \) attains the strong minimum (in a unique point).

In particular, the function \( \psi \) is bounded from below.

Let \( X \) be a dense subspace of a space \( Z \). By \( C(Z|X) \) we denote the space \( C(Z) \) of continuous bounded functions on \( Z \) with the norm \( \|g\| = \sup\{|g(x)| : x \in X\} \). We put \( C(X|X) = C(X) \). For any property \( \mathcal{P} \) of subsets of the space \( X \) and any bounded from below function \( f : X \to \mathbb{R}^\infty \) we put \( S_{\mathcal{P},Z}(f) = \{g \in C(Z) : M_X(f + g) \in \mathcal{P}\} \) and \( S_{M,Z}(f) = \{g \in C(Z) : M_X(f + g) \neq \emptyset\} \).

We consider that \( f + g \) is a function on \( X \) and \( (f + g)(x) = f(x) + g(x) \) for each \( x \in X \). In this case the set \( M_X(f + g) \) is closed in \( X \) for each \( g \in C(Z) \). The space \( C(Z|X) \) is a Banach algebra of bounded continuous functions.

The following questions are natural:

- \( P1 \). Under which conditions the set \( S_{\mathcal{P},Z}(f) \) is non-empty?

- \( P2 \). Under which conditions the set \( S_{\mathcal{P},Z}(f) \) is dense in the space \( C(Z|X) \)?

- \( P2 \). Under which conditions the set \( S_{\mathcal{P},Z}(f) \) contains a dense \( G_\delta \)-subset of the space \( C(Z|X) \), i.e. is a generic set?

For the property \( \mathcal{P} = \theta \) the Problem \( P2 \) was aroused by A. N. Tychonoff in [58].

These questions are typical for the distinct variational principles in optimization (see [39, 30, 31, 55, 12, 27]). Some optimization problems in topological spaces were studied in [17, 20, 21, 22, 23, 52, 55]. One of the main results in [39, 23] affirm that the set \( S_{\mathcal{P},Z}(f) = \{g \in C(X) : M_X(f + g) \neq \emptyset\} \) is dense in \( C(X) \) for each bounded from below lower semi-continuous proper function \( f : X \to \mathbb{R}^\infty \).

Our aim is to examine the above questions for some concrete properties of sets and sequences of sets.

Denote by \( LC(X) \) the space of all bounded from below lower semi-continuous proper functions \( f : X \to \mathbb{R}^\infty \).

The next statement is well known and permit to reduce the study of functions \( LC(X) \) to the case of compact spaces.
Proposition 8.6. Let \( X \) be a subspace of a space \( Z \). Then there exists a unique operator \( e_Z : LC(X) \to LC(Z) \) such that:

1. \( f = e_Z(f)|X \) and \( M_Y(f) = Y \cap M_Z(e_Z(f)) \) for any subspace \( Y \subseteq X \) and any function \( f \in LC(X) \);

2. If \( g \in LC(Z) \) and \( g|X = f \), then \( g \leq e_Z(f) \).

Proof. Let \( \psi : X \to \mathbb{R}^\infty \) be a function. We put \( \Psi_\infty = \Phi_\infty = Z \) and \( \Psi_t = \psi^{-1}(-\infty, t) \), \( \Phi_t = \cap \{ cl_Z \Psi_r : t < r \} \) for each \( t \in \mathbb{R} \). Then \( \varphi(z) = \inf \{ t : z \in \Phi_t \} \) is the lower semi-continuous regularization of the function \( \psi \) in the space \( Z \). If \( \psi \in LC(X) \), then we put \( e_Z(\psi) = \varphi \).

If \( g \in LC(Z) \) and \( g(x) \leq \psi(x) \) for each \( x \in X \), then \( g^{-1}(-\infty, t] \subseteq \psi^{-1}(-\infty, t] \) for each \( t \in \mathbb{R}^\infty \). Thus the function \( \varphi \) has the following properties:

1. If \( \psi \) is lower semi-continuous on \( X \), then \( \psi = \varphi|X \).

2. \( \varphi(z) = +\infty \) for each \( z \in Z \setminus cl_Z X \).

3. If \( g \in LC(Z) \) and \( g(x) \leq \psi(x) \) for each \( x \in X \), then \( g \leq \varphi \).

Let \( X = Z \). Fix \( f \in C(X) \). Denote by \( A_t \) and \( B_t \) the level sets of the functions \( \psi + f \) and \( \varphi + f \) respectively. Obviously, \( A_t \subseteq B_t \). We affirm that \( B_t = \cap \{ cl_X A_r : r \in (t, +\infty) \} \). Assume that \( y \notin \cap \{ cl_X A_r : r \in (t, +\infty) \} \). There exists \( r > t \) such that \( y \notin cl_X A_r \). Let \( 2s = \psi(y) + f(y) - r > 0 \). For any \( \delta \in (0, s) \) there exists an open subset \( U \) of \( X \) such that \( \psi(x) < f(x) + r \) and \( f(x) < f(y) + \delta \) for each \( x \in U \). Then \( \psi(x) > r - f(x) > r - f(y) - \delta \) for each \( x \in U \). Hence \( \varphi(x) \geq r - f(y) - \delta \) and \( \varphi(y) \geq r - f(y) - \delta \) for each \( x \in U \). Since \( \delta > 0 \) is arbitrary, \( \varphi(y) + f(y) \geq r \). Therefore \( B_t = \cap \{ cl_X A_r : r \in (t, +\infty) \} \) and the function \( \varphi + f \) is the lower semi-continuous regularization of the function \( \psi + f \).

Hence the function \( \varphi \) has the following properties too:

4. \( \inf_X (\varphi + f) = \inf_X (\psi + f) \) for any continuous function \( f \in C(X) \).

5. \( m_X(\varphi + f) = m_X(\psi + f) = m_Z(\psi + f) = M_Z(\varphi + f) \) for any continuous function \( f \in C(Z) \).

6. For any continuous function \( f \in C(Z) \) any minimizing sequence of the function \( \psi + f \) is a minimizing sequence of the function \( \varphi + f \).

The first variational principle of this kind seems to be the famous Bishop-Phelps theorem (see [11]): The set of continuous linear functionals in a real Banach space \( B \) attaining their infimum on a closed bounded convex set \( X \subset B \) is dense in the dual Banach space \( B^* \). Other examples are the Ekeland variational principle [30, 31], Stegall variational principle [55, 56], the smooth variational principles of Borwein and Preiss [12] and of Deville, Godefroy and Zizler [27, 28], as well as the "continuous" variational principle considered in Lucchetti and Patrone [45], De Blasi and Myiak [26] and [18, 20] (in which \( f \) is a continuous function). If \( X \) is a complete metric space one can frequently (but not always) show that the set \( S_X(f) = \{ g \in C(X) : f + g \text{ attains its infimum in } X \} \) contains a dense \( G_\delta \)-subset of \( C(X) \). In such a case the corresponding variational principle is called Generic variational principle.
9 Perturbations of functions

Fix a dense subspace $X$ of a compact space $Z$, i.e. $Z$ is a compactification of the space $X$.

Let $f : X \to \mathbb{R}^\infty$ be a proper bounded from below function and $\mathcal{P} \in \{k, \theta\}$.

In this section, using the methods developed in [39], we give the conditions under which the set $S_{\mathcal{P}, Z}(f)$ is dense in the space $C(Z|X)$.

Denote by $T$ the topology of the space $X$. Let $Y = dom(f)$. By $T_f$ we denote the topology on $X$ generated by the open base $\{U \cap f^{-1}(-\infty, r) : U \in T_X, r \in \mathbb{R}\}$. Let $X_f$ be the set $X$ with the topology $T_f$.

The set $Y = dom(f)$ is an open $F_\sigma$-set in the space $X_f$. Let $L_f = \{H \subseteq Y : H \in T_f\}$.

The family $L_f$ is the topology on $Y = dom(f)$ as a subspace of the space $X_f$. Let $Y_f$ be the subset $Y = dom(f)$ with the topology $L_f$.

**Definition 9.1.** We say that $X$ is a space with separating family in $Z$ if there is given a set $Sep(Z|X)$ with the next properties:

1. $Sep(Z|X) \subseteq C(Z|X)$ and $\|h\| \leq 1$ for any $h \in Sep(Z|X)$;
2. $\lambda h \in Sep(Z|X)$ for all $\lambda \in [-1, 1]$ and $h \in Sep(Z|X)$;
3. for any open subset $U \subseteq Z$, any point $x \in X$ and any $\epsilon > 0$ there exists $f \in Sep(Z|X)$ such that $f(x) = 1$ and $f(x) = 0$ for each $x \in X \setminus U$.
4. $Sep(Z|X)$ is closed in $C(Z|X)$ and $2^{-1}(f + g) \in Sep(X)$ for all $f, g \in Sep(Z|X)$. In this case $tf + (1-t)g \in Sep(Z|X)$ provided $f, g \in Sep(Z|X)$ and $t \in I$.

**Lemma 9.2.** Let $f : X \to \mathbb{R}^\infty$ be a proper bounded from below lower semi-continuous function on a space $X$ with a separating family $Sep(Z|X)$, $1 \geq \epsilon > 0$, $U$ be an open subset of the space $Z$, the set $V = U \cap X$ is non-empty and $m_V(f) < m_X(f) + \epsilon$.

Then there exists $g \in Sep(Z|X)$ such that:

1. $\|g\| < \epsilon$.
2. $cl_Z M_X(f + g) \subseteq U$ and $M_X(f + g) \subseteq V$.
3. There exists a real number $r$ such that $cl_Z \{x \in X : f(x) + g(x) < r\}$ is a non-empty subset of the set $U$.

**Proof.** Fix $x_0 \in V$ such that $f(x_0) < m_X(f) + \epsilon$. We put $s = f(x_0) - m_X(f)$.

Let $0 < 2\delta < m_X(f) + \epsilon - f(x_0) = \epsilon - s$.

**Case 1.** $s = 0$.

Fix an open subset $W$ of $Z$ and $\beta > 0$ such that $3\beta < \delta$ and $x_0 \in W \subseteq cl_Z W \subseteq U$. Obviously, $f(x_0) < \beta + m_X(f)$.

Fix a function $h \in Sep(Z|X)$ such that $\|h\| \leq 1$, $h(x_0) > 1 - \beta$ and $h(z) < \beta$ for any $z \in Z \setminus W$.

Put $g = -ch$. We have $f(x_0) + g(x_0) < f(x_0) - \epsilon(1 - \beta)$. Since $1 - \beta > 1 - \delta > \delta$, we have $f(x) + g(x) > f(x) - \epsilon(1 - \beta) > \inf f(x) - \epsilon(1 - \beta)$ for any $x \in Z \setminus W$. Let $f(x_0) + g(x_0) < f(x_0) - \epsilon(1 - \beta)$. Hence $x_0 \in \{x \in Z : f(x) + g(x) < r\} \subseteq W$. The set $\{z \in Z : e_Z(f + g)(z) \leq r\}$ is closed in $Z$ and $\{z \in Z : e_Z(f + g)(z) \leq r\} \subseteq U$. 

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Case 2. $s > 0$.

Obviously, $s < \epsilon$. Fix $\beta > 0$ such that $3\beta < \min\{\delta, s, \epsilon - s\}$. The set $F = \{x \in Z : e_Z(f)(x) \leq f(x_0) - \beta\}$ is closed in $Z$ and $x_0 \not\in F$. There exists an open subset $W$ of $Z$ such that $x_0 \in W \subseteq cl_Z W \subseteq U \setminus F$. Fix $h \in Sep(X)$ such that $\|h\| \leq 1$, $h(x_0) > 1 - \beta$ and $h(x) < \beta$ for any $x \in Z \setminus W$.

Let $g = -eh$. Then $f(x_0) + g(x_0) < f(x_0) - \epsilon(1 - \beta)$.

If $x \in X \setminus W$, then $f(x) + g(x) \geq f(x_0) - s - \epsilon \beta$.

Since $\epsilon - \epsilon \delta > s + \epsilon \delta$ we can fix a real number $r$ such that $f(x_0) - \epsilon(1 - \beta) < r < f(x_0) - s - \epsilon \beta$.

Hence $x_0 \in \{x \in X : f(x) + g(x) < r\} \subseteq W$.

The proof is complete. 

Let $\mathcal{P}$ be a property of subsets and sequences of subsets. A space $X$ has the property $\mathcal{P}$ in the point $x \in X$ if there exists a convergent sequence $\{U_n : n \in \mathbb{N}\}$ on $X$ of open subsets of a space $Z$ such that $x \in \bigcap\{U_n : n \in \mathbb{N}\}$ and any closed subset $F \subseteq \bigcap\{U_n : n \in \mathbb{N}\}$ of the space $X$ has the property $\mathcal{P}$.

The proof of the next assertion is similar with the proof of Lemma 2.1 from [39] and may be considered as the one of topological generalizations of the Ekeland’s [30, 14, 46] variational principle for topological spaces.

**Theorem 9.3.** Let $f : X \to \mathbb{R}^\infty$ be a proper bounded from below lower semi-continuous function on a space $X$ with a separating family $Sep(Z|X)$, $\epsilon > 0$, $\{U_n : n \in \mathbb{N}\}$ be a sequence of open subsets of the space $Z$, $x_0 \in \text{dom}(f) \cap (\bigcap\{U_n : n \in \mathbb{N}\})$ and $f(x_0) < \epsilon + m_X(f)$.

Then there exists a Cauchy series $\{g_n \in Sep(Z|X) : n \in \mathbb{N}\}$ such that:

1. $\|g_n\| < 2^{-n}\epsilon$ for each $n \in \mathbb{N}$.
2. If $g = \Sigma\{g_n : n \in \mathbb{N}\}$, then $\|g\| \leq \epsilon$ and $x_0 \in M_X(f + g) \subseteq \bigcap\{U_n : n \in \mathbb{N}\}$.
3. $f(x) > f(x_0) - g(x)$ for any $x \in X \setminus \bigcap\{U_n : n \in \mathbb{N}\}$.
4. If $\{x_n : n \in \mathbb{N}\}$ is a minimizing sequence of the function $f + g$, then for any $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $\{x_i : i \geq m\} \subseteq U_n \cap \{x \in X : f(x) < f(x_0) + 2^{-n}\}$.
5. The set $S_{M,Z}(f) = \{g \in C(Z|X) : M_X(f + g) \neq \emptyset\}$ is dense in $C(Z|X)$.
6. If $\mathcal{P}$ is a property of subsets and sequences of subsets of the space $X$, any closed subset $\Phi \subseteq \bigcap\{U_n : n \in \mathbb{N}\}$ of the space $X$ has the property $\mathcal{P}$, $\{W_n = \{x \in X : f(x) + g(x) < m(f + g) + 2^{-n} : n \in \mathbb{N}\} \text{ and } \{U_n \cap X : n \in \mathbb{N}\}$ is a convergent sequence with the property $\mathcal{P}$ in the point $x_0 \in X$, then:
   - $\Phi \cap \{W_n \cap X : n \in \mathbb{N}\}$ is a closed countably compact subset with the property $\mathcal{P}$ of the space $X$;
   - $\{W_n \cap X : n \in \mathbb{N}\}$ is a convergent sequence of of the open subsets of the spaces $X_f$ and $Y_f$;
   - if the set $W$ is open in $Y_f$ and $\Phi \subseteq W$, then $W_n \subseteq W$ for some $n \in \mathbb{N}$;
   - any minimizing sequence $\{x_n : n \in \mathbb{N}\}$ of the function $f + g$, has an accumulation point in the spaces $X$ and $Y_f$;
   - the function $f$ is constant on $\Phi$ and $\mathcal{T}|\Phi = \text{mathcal{T}}|\Phi$. 

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Proof. We can assume that $cl_Z U_{n+1} \subseteq U_n$ for each $n \in \mathbb{N}$. Moreover, let $\{V_n : n \in \mathbb{N}\}$ be a sequence of open subsets of $Z$ such that $x_0 \in cl_Z V_{n+1} \subseteq V_n \cap U_n$ for each $n \in \mathbb{N}$. The set $F = \cap \{V_n : n \in \mathbb{N}\}$ is closed in $Z$ and $x_0 \in F$.

We put $3\delta = \epsilon + m_X(f) - f(x_0)$, $s = f(x_0) - \inf_X(f)$ and $r = s + \delta$. Obviously, $\epsilon > r \geq \delta > 0$ and $f(x_0) < r + m_X(f)$.

Now we will use Lemma 9.2.

Case 1. $s = 0$.

For any $n \in \mathbb{N}$ fix a function $g_n \in Sep(Z|X)$ such that $g_n(x) = -2^{-n-1}\delta$ for any $x \in V_{n+2}$, $\| g_n \| = 2^{-n-1}\delta$ and $g_n(x) = 0$ for any $x \in Z \setminus V_n$.

Obviously, $\{g_n : n \in \mathbb{N}\}$ is a Cauchy series. Let $g = \Sigma\{g_n : n \in \mathbb{N}\}$. By construction, $g(x) = \Sigma\{g_n(x) : n \in \mathbb{N}\} = -\delta \ f(x_0) + g(x_0) = f(x_0) - \delta$ and $f(x) + g(x) = f(x) - \delta \geq f(x_0) - \delta$ for each $x \in F$.

Let $x \in Z \setminus F$. There exists $m \in \mathbb{N}$ such that $x \in X \setminus V_m$. Then $g_n(x) = 0$ for any $n \geq m$, $g(x) \geq \delta(1 - 2^{-m+1})$ and $f(x) + g(x) \geq f(x_0) - \delta(1 - 2^{-m+1}) > f(x_0) - \delta$.

If $\{x_n : n \in \mathbb{N}\}$ is a minimizing sequence of the function $f + g$, then:

- $\{x_n : n \in \mathbb{N}\}$ is a minimizing sequence of the function $f$ and $\lim_{n \to \infty} g(x_n) = -\delta$;
- for any $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $\{x_i : i \geq m\} \subseteq V_n \cap \{x \in X : f(x) < f(x_0) + 2^{-n}\}$.

By construction, we have:

- any sequence $\{x_n \in W_n : n \in \mathbb{N}\}$ is a minimizing sequence of the function $f + g$;
- $f(x) = f(x_0)$ and $f(x) + g(x) = f(x_0) + g(x_0) = f(x_0) - \delta$ for each $x \in \Phi$.

In this case the Cauchy series $\{g_n : n \in \mathbb{N}\}$ is constructed.

Case 2. $s > 0$.

Now we follow the proof of Lemma 2.1 from [39].

Obviously, the set $L_n = \{x \in Z : e_Z(f)(x) \leq m_X(f) + s(1 - 2^{-n})\}$ is closed in $Z$, $x_0 \notin L_n \subseteq L_{n+1}$ for each $n \in \mathbb{N}$ and $L = \cup\{L_n : n \in \mathbb{N}\} = \{x \in Z : e_Z(f)(x) < m_X(f) + s = f(x_0)\}$.

Let $\{V_n : n \in \mathbb{N}\}$ be a sequence of open subsets of $Z$ such that $x_0 \in cl_Z V_{n+1} \subseteq V_n \cap (U_n \setminus L_n)$ for each $n \in \mathbb{N}$. The set $F = \cap \{V_n : n \in \mathbb{N}\}$ is closed in $Z$ and $x_0 \in F$.

For any $n \in \mathbb{N}$ fix a function $p_n \in Sep(Z|X)$ such that $p_n(x_0) = 1$ and $(X \setminus V_{n+1}) \subseteq p_n^{-1}(0)$. Let $h_n = -2^{-n}r p_n$, $q_n = -2^{-n}\delta p_n$ and $g_n = h_n + q_n = -2^{-n}(r + \delta)p_n$. By construction, $h_n, q_n, g_n \in Sep(Z|X)$.

There exist the sums $h = \Sigma\{h_n : n \in \mathbb{N}\}$, $q = \Sigma\{q_n : n \in \mathbb{N}\}$, $g = \Sigma\{g_n : n \in \mathbb{N}\}$ and $g = h + q$.

We put $\psi = f + h$. By construction, $h(x_0) = -r$. Thus $\psi(x_0) = f(x_0) - r$.

If $x \in X \setminus V_1$, then $h_n(x) = 0$ for each $n \in \mathbb{N}$, $h(x) = 0$ and $\psi(x) = f(x) + h(x) = f(x) \geq f(x_0) - s > f(x_0) - r = \psi(x_0)$. Thus $M_Z(h) \subseteq cl_Z V_1 \subseteq U_1$.

Let $x \in V_1$. If $f(x) \geq f(x_0)$, then $\psi(x) \geq \psi(x_0)$. Assume that $f(x) < f(x_0)$. Then $x \in L$. Denote by $m$ the first natural number $n \in \mathbb{N}$ for which $x \in L_n$. If $m = 1$, then $h(x) = 0$ and $\psi(x) = f(x_0) - s > \psi(x_0)$.

Assume that $m > 1$. Then $x \in L_m \setminus L_{m-1}$. Thus $f(x) > \inf_X(f) + s(1 - 2^{-m+1})$, $p_n(x) = 1$ for any $n \geq m$ and $\psi(x) = f(x) + h(x) > \inf_X(f) + s(1 - 2^{-m+1})$. Thus $f(x) > \inf_X(f) + s(1 - 2^{-m+1})$.
$2^{-m+1} - r. \Sigma(2^{-n} : n \geq m) = \inf_{X}(f)f + (1 - 2^{-m+1}) - r. \Sigma(2^{-n} : n \geq m) > \inf_{X}(f)f + \delta = \psi(x_0)$. Hence $\psi(x) \geq \psi(x_0)$ for any $x \in V_1$. In particular, $x_0 \in M_X(\psi)$. Then, as in Case 1, we obtain that $x_0 \in M_{\beta X}(\psi + q) \subseteq N\{U_n : n \in \mathbb{N}\}$. Since $f + g = \psi + q$, the proof is complete. \hfill \Box

If $P$ is a property of subsets and sequences of subsets of a space $X$, then we put $X(P) = \{ x \in X : X \text{ has the property } P \text{ in the point } x \}$.

The properties of the set $S_{(\mathcal{P}, Z)}(f)$ are determined by the topological properties of the subspace $\text{dom}(f)$.

Example 9.4. Let $Y$ be a closed subspace of a space $X$. Consider the function $f : X \to \mathbb{R}^\infty$, where $f(x) = 0$ for $x \in Y$ and $f(x) = 1$ for $x \in X \setminus Y$. By construction, $f$ is a proper bounded from below lower semi-continuous function on a space $X$. Assume that $Y$ is a space without points of countable pseudocharacter and $X$ contains some dense complete metrizable subspace $S$. If $g$ is a continuous function on $X$ and $\|g\| < 1$, then $M_X(f + g)$ is not a singleton. Thus the set $\{g \in C(Z|X) : M_X(f + g) \text{ is a singleton} \}$ is dense in $C(Z|X)$.

A point $x \in Y$ is of the countable pseudocharacter in $Y$ if $\{x\}$ is a $G_\delta$-subset of $Y$.

Corollary 9.5. Let $f : X \to \mathbb{R}^\infty$ be a proper bounded from below lower semi-continuous function on a space $X$. Assume that in any non-empty closed in $X$ subspace $Y \subseteq \text{dom}(f)$ some point is of countable pseudocharacter in $Y$. Then the set $\{g \in C(Z|X) : M_X(f + g) \text{ is a singleton} \}$ is dense in $C(Z|X)$.

A set $F$ of a space $Y$ is of the countable character in $Y$ if there exists a countable family $\gamma$ of open subsets of $Y$ such that for each open set $U$ that contains $F$ we have $F \subset V \subset U$ for some $V \in \gamma$.

Corollary 9.6. Let $f : X \to \mathbb{R}^\infty$ be a proper bounded from below lower semi-continuous function on a space $X$. Assume that in any non-empty closed in $X$ subspace $Y \subseteq \text{dom}(f)$ some point is of countable character in $Y$. Then the set $S_{(\mathcal{Q}, Z)}(f)$ is dense in $C(Z|X)$.

Corollary 9.7. Let $f : X \to \mathbb{R}^\infty$ be a proper bounded from below lower semi-continuous function on a space $X$. Assume that in any non-empty closed in $X$ subspace $Y \subseteq \text{dom}(f)$ some non-empty compact has countable character in $Y$. Then the set $S_{(\mathcal{K}, Z)}(f)$ is dense in $C(Z|X)$.

10 General properties of the mapping $M_{(X,f)}$

Fix a space $X$, a proper bounded from below lower semi-continuous function $f : X \to \mathbb{R}^\infty$ on a space $X$ and a compactification $Z$ of the space $X$.

Consider the set-valued mapping $M_{(X,f)} : C(Z|X) \to Y$, where $M_{(X,f)} = M_X(f + g)$ for each $g \in C(Z|X)$.

In this section, using the methods developed in [20, 21, 22, 23], we study the structure of the set-valued mapping $M_{(X,f)}$ and the general properties of the set $S_{(\mathcal{Q}, Z)}(f)$.
For any \( r > 0 \) and \( g \in C(Z|X) \) we put \( M_{(X,f,r)}(g) = \{x \in X : f(x) + g(x) \leq r + m_X(f + g)\} \). We have \( M_{(X,f)}(g) = \cap \{M_{(X,f,r)}(g) : r > 0\} \). Obviously, \( M_{(X,f)}(g) \subseteq M_{(X,f,r)}(g) \) for each \( r > 0 \).

The mapping \( M_{(X,f)} \) is called the solution mapping of the continuous perturbations of the function \( f \), the mapping \( M_{(X,f,r)} \) is the \( r \)-solution mapping of the continuous perturbations of the function \( f \), where \( r \geq 0 \). If \( M_{(Z,f)}(g) = \cap \{cZM_{(X,f,r)}(g) : r > 0\} \), then \( M_{(Z,f)}(g) = M_{(Z,e_X(f))}(e_X(g)) \), where \( e_X : LC(X) \rightarrow LC(Z) \) is the extension operator described in Proposition 8.6.

We may to say that \( M_{(Z,f)}(g) \setminus M_{(X,f)}(g) \) is the set of the \( Z \)-virtual solutions of the minimization problem \((X, f + g)\).

We say that the minimization problem \((X, f + g)\) is \( Z \)-correct if \( cZM_{(X,f)}(g) = M_{(Z,f)}(g) \).

If \( L \) and \( M \) are two subsets of the space \( X \), then we put \( L \ll M \) if there exists a continuous function \( h \in C(Z|X) \) such that \( L \subseteq h^{-1}(0) \) and \( X \setminus M \subseteq h^{-1}(1) \).

For any subset \( H \) of \( X \) we put \( M^{\approx}_{(X,f)}(H) = \{g \in C(X) : M_{(X,f,r)}(g) \ll H \text{ for some } r > 0\} \) and \( M^{\approx}_{(X,f)}(H) = \{g \in C(X) : M_{(X,f,r)}(g) \ll H \text{ for some } r > 0\} \).

Obviously, \( M^{\approx}_{(X,f)}(H) \subseteq M^{\approx}_{(X,f)}(H) \).

We say that the mapping \( M_{(X,f)} \) is \( \delta \)-upper semi-continuous if the set \( M^{\approx}_{(X,f)}(H) \) is open in \( C(Z|X) \) for each open subset \( H \) of \( X \).

We say that the mapping \( M_{(X,f)} \) is \( \lambda \)-upper semi-continuous if the set \( M^{\approx}_{(X,f)}(H) \) is open in \( C(Z|X) \) for each open subset \( H \) of \( Y_f \).

The graph of a \( \delta \)-upper semi-continuous mapping \( M_{(X,f)} \) is closed in \( C(Z|X) \times X \) (see Proposition 2.4 from [23]).

**Theorem 10.1.** Let \( Z \) be a compactification of a space \( X, f \in LC(X) \) and \( Y_f \) be the set \( Y = \text{dom}(f) \) with the topology \( \mathcal{L}_f \). Then the next assertions are true:

1. The mapping \( M_{(X,f)} : C(Z|X) \rightarrow X \) is \( \delta \)-upper semi-continuous and is \( \lambda \)-upper semi-continuous.
2. If \( U \) is an open non-empty subset of the space \( Y_f \), then \( M^{\approx}_{(X,f)}(U) \neq \emptyset \).
3. If \( U \) is an open subset of \( Z \) and \( U \cap Y \neq \emptyset \), then \( M^{\approx}_{(X,f)}(U) \neq \emptyset \).
4. The set \( M^{\approx}_{(X,f)}(U) \) is dense in the set \( M^{-1}_{(Y,f)}(U) \) for any open subset \( U \) of \( Y_f \).
5. The set \( M^{\approx}_{(X,f)}(U) \) is dense in the set \( M^{-1}_{(Z,f)}(U) \) for any open subset \( U \) of \( Z \).
6. \( M_{(X,f)} \) is an open mapping of the space \( C(Z|X) \) onto the space \( Y_f \).
7. If the function \( f_x = f|\text{dom}(f) \) is continuous on \( \text{dom}(f) \), then \( M_{(X,f)} \) is an open mapping of \( C(Z|X) \) onto \( Y \).
8. \( \text{Dom}(M_{(X,f)}) = \{g \in C(X) : M_{(X,f)}(g) \neq \emptyset \} \) is dense in \( C(Z|X) \).
9. If the set \( L \subseteq M_{(X,f)}(f) \) is dense in \( C(Z|X) \), then the set \( M_{(X,f)}(f) \) is dense in the space \( Y_f \).
10. If the set \( L \subseteq Y \) is dense in \( Y_f \), then the set \( M_{(X,f)}(f) \) is dense in the space \( M_{(X,f)}(f) \).

**Proof.** Let \( U \) be an open non-empty subset of the space \( Y_f \). Fix \( g \in C(Z|X) \) and a point \( y_0 \in M_{(X,f)}(g) \). There exist an open subset \( W \) of the compactification \( Z \) of the space \( X \) and a real number \( r \) such that \( y_0 \in W \cap f^{-1}(-\infty, r) \subseteq U \).
We consider that $0 < 4\delta < r - f(y_0)$. The set $F = \{ x \in X : f(x) \leq f(y_0) - \delta \}$ is closed in $X$. There exists a continuous function $h : Z \rightarrow [0, \epsilon]$ such that $h \in C(Z|X)$, $h(y_0) = 0$ and $(Z \setminus W) \cup F \cup \{ z \in Z : |g(z) - g(y_0)| < \delta \} \subseteq h^{-1}(2\delta)$. Let $g_1 = g + h$. Then $\| g_1 - g \| = 2\delta$ and $y_0 \in M_{f(x)}(g_1) \subseteq W \setminus F$. Assume that $\varphi \in C(Z|X)$ and $\| \varphi - g_1 \| < \delta$. Then $\varphi(y_0) \leq g(y_0) + \delta$ and $m_X(f + \varphi) \leq f(y_0) + \varphi(y_0) < f(y_0) + g(y_0) + \delta = m_X(f + g) + \delta$. Hence, $m_X(f + \varphi) < m_X(f + g) + \delta$. Let $\lambda(\varphi) = (m_X(f + g) + \delta) - m_X(f + \varphi)$. Fix $y \in Y \setminus U$. We affirm that $f(y) + \varphi(y) \geq m_X(f + g) + \delta$.

**Case 1.** $f(y) \geq r$.

In this case $f(y) > f(y_0) + 4\delta$, $\varphi(y) \geq g_1(y) - \delta \geq g(y) - 2\delta - \delta = g(y) - 3\delta \geq g(y_0) - 3\delta$ and $f(y) + \varphi(y) > f(y_0) + \varphi(y_0) + 4\delta + g(y_0) - 3\delta = m_X(f + g) + \delta$.

**Case 2.** $y \in F$.

In this case $f(y) \leq f(y_0) - \delta$, $f(y) + g(y) \geq m_X(f + g)$, $g_1(y) = g(y) + 2\delta$, $\varphi(y) \geq g(y) + \delta = g(y) + \delta$ and $f(y) + \varphi(y) \geq f(y) + g(y) + \delta \geq m_X(f + g) + \delta$.

**Case 3.** $y \in Y \setminus W$.

In this case $g_1(y) = g(y) + 2\delta$, $\varphi(y) = g(y) + \delta$ and $f(y) + \varphi(y) > f(y) + g(y) + \delta \geq m_X(f + g) + \delta$.

Therefore, $f(y) + \varphi(y) > g_1(y) - \delta = g(y) + \delta$ and $f(y) + \varphi(y) > f(y) + g(y) + \delta \geq m_X(f + g) + \delta$.

Thus, $m_X(f + g) + \delta = \lambda(\varphi)$ for each $y \in Y \setminus U$ and $\lambda(\varphi) > 0$ provided $\| \varphi - g_1 \| < \delta$. In particular, we have $M_{f(x)}(\varphi) \subseteq W \cap f^{-1}(-\infty, r) \subseteq U$ and $\{ y \in Y : f(y) + \varphi(y) < m_X(f + \varphi) + \lambda(\varphi) \} \subseteq U$.

If the sets $U$ and $W$ are open in $Z$ and $cl_Z W \supseteq U$, then $M_{f(x)}(\varphi) \subseteq M_{f(x)}(\varphi)(U)$.

The assertions 1 - 4 are proved.

Fix an open subset $U$ of the space $Z$. Suppose that the set $U \cap Y \neq \emptyset$. Fix $g \in M_{f(x)}(U)$, $\epsilon > 0$ and $x_1 \in U \cap M_{f(x)}(f + g)$. There exists an open subset $V$ of $Z$ such that $x_1 \in V \subseteq U$. Obviously, $f + g \in LC(X)$ and $x_1 \in dom(f + g) = dom(f)$ for each $g \in C(Z|X)$. There exists a continuous function $h : Z \rightarrow [0, \epsilon]$ such that $h \in C(Z|X)$, $h(x_1) = 0$ and $X \setminus V \subseteq h^{-1}(\epsilon)$. Let $\varphi = g + h$. Then $\| \varphi - g \| < \epsilon$ and $M_{f(x)}(f + \varphi) \subseteq V \subseteq U$. Hence $\varphi \in M_{f(x)}(U)$. The assertion 5 is proved.

The assertion 8 follows from Theorem 9.3.

Let $U$ be an open non-empty subset of the space $C(Z|X)$. Obviously, the function $f_U = f|Y$ is continuous on $Y_f$. From Theorem 9.3 (see Corollary 2.2 from [23]) it follows that the set $M_{f(x)}(U)$ is non-empty. Fix $x_0 \in M_{f(x)}(U)$. There then exist $\delta > 0$ and $g \in C(Z|X)$ such that $x_0 \in M_X(f + g)$ and $\{ b \in C(X) : \| g - b \| < 4\epsilon \} \subseteq U$. Let $V_1 = \{ x \in Y : f(x) < f(x_0) + \epsilon \}$, $V_2 = \{ z \in Z : |g(z) - g(x_0)| < \epsilon \}$, $F = \{ x \in X : f(x) \leq f(x_0) - \epsilon \}$, $V_3 = V_2 \cap (Z \setminus cl_Z F)$ and $V = V_1 \cap V_2 \cap (Z \setminus cl_Z F) = V_1 \cap V_3$. The set $V_1$ is open in $Y_f$, the set $V_2$ is open in $Z$, the set $F$ is closed in $X$ and the set $V_3$ is open in $Z$. Thus the set $V$ is open in $Y_f$ and $x_0 \in V$. Fix $x_1 \in V$. By construction, $\varphi = f + g \in LC(X)$ and $\varphi(x_1) < m_X(\varphi) + 2\epsilon = f(x_0) + g(x_0) + 2\epsilon$. Since $x_1 \in V_3$ and the set $V_3$ is open in $Z$, by virtue of Theorem 9.3, there exists $h \in C(Z|X)$ such that $\| h \| < 2\epsilon$ and $x_1 \in M_X(\varphi + h)$. Let $g_1 = g + h$. Then $\| g - g_1 \| < 2\epsilon$, $g_1 \in U$ and $x_1 \in M_{f(x)}(g_1)$. Thus $x_0 \in V \subseteq M_{f(x)}(U)$. Assertion 6 is proved.
Assertion 7 is proved in [23] and follows from Assertion 6. Assertion 9 follows from Assertions 1 and 2.

Fix \( g \in C(Z|X) \) and \( \epsilon > \delta > 0 \). Let \( V = \{ y \in Y: f(y) + g(y) < m_Y(f + g) + \delta \} \). By construction, \( V \) is an open non-empty subset of the space \( Y_f \). Then \( V \cap W \neq \emptyset \). Fix a point \( x_0 \in V \cap W \). There exists an open subset \( U \) of the space \( Z \) and \( r \in \mathbb{R} \) such that \( x_0 \in U \cap f^{-1}(-\infty, r) \subseteq V \cap W \). By virtue of Theorem 9.3, there exists a function \( h \in C(Z|X) \) such that \( ||h|| \leq \delta \) and \( x_0 \in M_X(f + g + h) \). Put \( \varphi = g + h \). Then \( \varphi \in C(Z|X) \), \( ||\varphi - g||_X < \epsilon \) and \( x_0 \in M_X(f + \varphi) = M_{(X,f)}(\varphi) \). Thus \( M_{(X,f)}(\varphi) \cap W \neq \emptyset \) and \( \varphi \in M_{(X,f)}^{-1}(W) \). Assertion 10 is proved. The proof is complete.

Theorem 10.2. Let \( \mathcal{P} \) be a property of subsets and sequences of subsets of a space \( X \) and \( f: X \to \mathbb{R}^\infty \) be a proper bounded from below lower semi-continuous function on the space \( X \) with the following properties:

- if \( \{ U_n : n \in \mathbb{N} \} \) is a sequence with the property \( \mathcal{P} \), then it is convergent in \( X \) and any closed subset \( \Phi \subseteq \cap \{ U_n : n \in \mathbb{N} \} \) of the space \( X \) has the property \( \mathcal{P} \);
- the set \( Y \cap X(\mathcal{P}) \) is dense in \( Y_f \).

Then:
1. The set \( S_{(\mathcal{P},Z)}(f) \) is dense in \( C(Z|X) \).
2. The mapping \( M_{(X,f)}: S_{(\mathcal{P},Z)}(f) \to X \) is upper semi-continuous.
3. The mapping \( M_{(X,f)}: S_{(Z)}(f) \to Y_f \) is open and upper semi-continuous.
4. If \( L \) is an dense subset of the space \( S_{(Z)}(f) \), then \( M_{(X,f)}(L) = \cup \{ M_{(X,f)}(g) : g \in C(Z|X) \} \) is dense in \( Y_f \).
5. The set \( \{ g \in C(Z|X) : M_{(X,f)}(g) \subseteq U \} \) is dense in the set \( M_{(Z)}^{-1}(U) \) for any open subset \( U \) of \( Y_f \).
6. If the set \( L \subseteq Y \cap X(\mathcal{P}) \) is dense in \( Y_f \), then the set \( M_{(X,f)}^{-1}(L) \cap S_{(\mathcal{P},Z)}(f) \) is dense in the space \( M_{(X,f)}(f) \).

Proof. It is similar to the proof of Theorem 10.1.

Corollary 10.3. Let \( X \) be a space, \( f \in LC(X) \), \( S_{(\mathcal{P},Z)}(f) \) be dense in \( C(Z|X) \), \( G \) be the graphic in \( X \times S_{(\mathcal{P},Z)}(f) \) of the mapping \( M_{(X,f)}: S_{(\mathcal{P},Z)}(f) \to X \), \( \varphi: G \to X \) and \( \psi: G \to S_{(\mathcal{P},Z)}(f) \) be the natural projections, \( \mathcal{P} \{ k, \theta \} \) be a property of subsets and sequences of subsets of spaces.

Then:
1. The mapping \( \varphi \) is open and continuous.
2. The mapping \( \psi \) is closed, continuous and all its fibers are compact subspaces with the property \( \mathcal{P} \).
3. If \( S_{(k,Z)}(f) \) contains a dense \( G^\delta \)-subspace of the space \( C(Z|X) \), then \( G \) and \( Y_f \) contain dense \( \check{C}ech \)-complete subspaces.
4. If \( S_{(g,Z)}(f) \) contains a dense \( G^\delta \)-subspace of the space \( C(Z|X) \), then \( G \) and \( Y_f \) contain dense completely metrizable subspaces.

Proof. If \( A \subseteq X \) and \( B \subseteq C(Z|X) \), then \( \varphi(G \cap (A \times B)) = A \cap M_{(\mathcal{P},Z)}(B) \). Thus the mapping \( \varphi \) is open and continuous.
Example 10.4. Let \( W \) be an open subset of the space \( X \times C(Z|X) \), \( g \in S_{(Y,Z)}(f) \) and \( \psi^{-1}(g) \subseteq W \). Fix a countable base \( \{ V_n : n \in \mathbb{N} \} \) of the space \( C(Z|X) \) at the point \( g \). For each \( n \in \mathbb{N} \) we put \( W_n = \bigcup \{ W : W \text{ is open in } X, W \times V_n \subseteq W \} \). Since the set \( \psi^{-1}(g) \) is compact, there exists \( n \) for which \( W_n \times V_n \subseteq W \). Thus the mapping \( \psi \) is closed, continuous and all its fibers are compact subspaces with the property \( P \).

Let \( S \subseteq S_{(Y,Z)}(f) \) be a dense \( G_\delta \)-subset of the space \( C(Z|X) \). Then \( G_1 = \psi^{-1}(S) \) is a \( \check{C} \)-complete \( G_\delta \)-subset of a space \( G \). By virtue of Assertion 9 of Theorem 10.1, the set \( \varphi(G_1) \) is dense in the space \( Y_f \). Thus \( Y_f \) contains a dense \( \check{C} \)-complete subspace.

Example 10.5. Let \( Y \) be the subspace of rational numbers of the space \( X = \mathbb{Z} = I \). Consider that \( Y = \{ r_n : n \in \mathbb{N} \} \), \( f(r_n) = n \) and \( f(x) = +\infty \) for all \( n \in \mathbb{N} \) and \( x \in X \setminus Y \). Then:

1. The set \( S_{(g,Z)}(f) \) contains a dense \( G_\delta \)-subset of \( C(Z|X) \).
2. The space \( Y_f \) is complete metrizable and discrete.
3. The space \( Y \) does not contain a dense complete metrizable subspace.
4. \( M_{(X,f)}(g) \) is a non-empty finite subset for each \( g \in C(X) \).

11 Topological games

Let \( X \) be a topological space. Each topological game in \( X \) is described by two types of rules. The \textit{playing rules} that determine how to play the game and the \textit{winning rule} which determines the winner. In the topological games we consider below the playing rules coinciding with the playing rules of the classical \textit{Banach-Mazur game} \( G_{BM}(X) \). The winning rule differs from game to game and, actually, identifies the game.

Let us first recall the playing rules of the games of the Banach-Mazur type, where two players, \( \Sigma \) and \( \Omega \), select (choose) alternatively non-empty subsets of \( X \). The player \( \Sigma \) begins the game by selecting subset \( A_1 \) of the space \( X \). Then the player \( \Omega \) responds by selecting some non-empty subset \( B_1 \) of the subspace \( A_1 \). In turn, player \( \Sigma \) picks up a non-empty subset \( A_2 \) of \( B_1 \). On the \( n \)-th stage the player \( \Sigma \) chooses some non-empty set \( A_n \) containing in the previous choose set \( B_{n-1} \) of \( \Omega \) and \( \Omega \) chooses a non-empty subset \( B_n \) of the subspace \( A_n \). Continuing the game in this way the two players generate a nested infinite sequence \( \{ A_n, B_n : n \in \mathbb{N} \} \) of non-empty subsets of \( X \). The sets \( A_n \) are selected
by the player Σ and the sets \( B_n \) are selected by the player Ω. The sequence \( \{A_n, B_n : n \in \mathbb{N}\} \) is called a play.

In the Banach-Mazur game the sets \( B_n \) and \( A_n \) selected by the players Ω and Σ are open in \( X \) (see [57, 7, 38, 59]).

Let \( \mathcal{P} \) be any property of sequences \( \{H_n : n \in \mathbb{N}\} \) of non-empty subsets of topological spaces, \( \mathcal{F} \) be a family of subsets of a space \( X \) and \( \mathcal{F} \) be the topology on \( X \).

We consider that each sequence \( \{H_n : n \in \omega\} \) of the space \( X \) with the property \( \mathcal{P} \) satisfies the following conditions:

- the sequence \( \{H_n : n \in \omega\} \) is monotone: \( H_{n+1} \subseteq H_n \) for each \( n \in \mathbb{N} \);
- if \( \{L_n : n \in \mathbb{N}\} \) is a monotone sequence of subsets and \( cl_X L_n = cl_X H_n \) for each \( n \in \mathbb{N} \), then the sequence \( \{L_n : n \in \mathbb{N}\} \) has the property \( \mathcal{P} \);
- if \( \{L_n : n \in \mathbb{N}\} \) is a sequence of subsets and for each \( n \in \mathbb{N} \) there exists \( m > n \) such that \( H_m \subseteq L_{n+1} \subseteq L_n \) and \( L_m \subseteq H_n \), then the sequence \( \{L_n : n \in \omega\} \) has the property \( \mathcal{P} \).

The property \( \mathcal{P} \) and the families \( \mathcal{F} \) and \( \mathcal{L} \) of a space \( X \) determine on \( X \) a game \( G_{\mathcal{P}, \mathcal{F}, \mathcal{L}}(X) \): \( p = \{A_n, B_n : n \in \mathbb{N}\} \) is a play in the game \( G_{\mathcal{P}, \mathcal{F}, \mathcal{L}}(X) \) if \( A_n \in \mathcal{F} \) and \( B_n \in \mathcal{L} \) for each \( n \in \mathbb{N} \).

**Definition 11.1.** The player Ω wins the play \( p = \{A_n, B_n : n \in \mathbb{N}\} \) in the game \( G_{\mathcal{P}, \mathcal{F}, \mathcal{L}}(X) \), if:

- the sequence \( \{cl_X A_n : n \in \mathbb{N}\} \) has the property \( \mathcal{P} \);
- \( \cap\{cl_X A_n : n \in \mathbb{N}\} \neq \emptyset \).

Otherwise the player Σ is said to have won this play.

We put \( G_{\mathcal{P}, \mathcal{F}, \mathcal{L}}(X) = G_{\mathcal{P}}(X) \) and \( G_{\mathcal{P}, \mathcal{F}, \mathcal{L}}(X) G_{\mathcal{P}}(X) \).

We will need also the notion of ”a strategy for a player”. This notion does not depend on the winning rule and will have one and the same meaning in all games which we consider.

By a strategy \( \omega \) for the player Ω we mean ”a rule” that specifies each move of the player Ω in every possible situation.

The play \( p = \{A_n, B_n : n \in \omega\} \) is said to be an \( \omega \)-play, if the moves of player Ω were made according to the strategy \( \omega \) and we consider that \( B_n = \omega(A_1, B_1, ..., A_n) \) for every \( n \in \mathbb{N} \) and \( A_1, B_1, ..., B_{n-1}, A_n \). A strategy \( \omega \) for the player Ω is called a winning strategy in the game \( G_{\mathcal{P}, \mathcal{F}, \mathcal{L}}(X) \), if the player Ω wins each \( \omega \)-play in the sense of the winning rule of the game \( G_{\mathcal{P}, \mathcal{F}, \mathcal{L}}(X) \). If a such strategy exists, then the topological space \( X \) is called \( \Omega \)-favorable.

By a strategy \( s \) for the player Σ we mean a ”a rule” that specifies each move of the player Σ in every possible situation.

The topological space \( X \) is called \( \Sigma \)-unfavorable, if the player Σ does not have a winning strategy in the game \( G \). This means that, for every strategy \( s \) of player Σ, there is a \( s \)-play which is won by Ω. Evidently, each \( \Omega \)-favorable space \( X \) is \( \Sigma \)-unfavorable. The inverse implication is not valid.

As shown in [57], the following statement has place.

**Theorem 11.2.** On a space \( X \) consider the game \( G_{\mathcal{P}}(X) \). Let \( \mathcal{P} \) be a property of sequences of sets and \( \cap\{cl_X A_n : n \in \mathbb{N}\} \neq \emptyset \) for each sequence \( \{A_n : n \in \omega\} \).
of a space $X$ with the property $\mathcal{P}$. If the space $X$ is $\Sigma$-unfavorable, then $X$ is a space with Baire property (i.e. the intersection of any countable family of open and dense subsets of $X$ is again dense in $X$).

Remark. Let $\mathcal{P}$ and $\mathcal{E}$ be two properties of sequences of non-empty sets. We consider that the property $\mathcal{P}$ is stronger than the property $\mathcal{E}$ if each sequence with the property $\mathcal{P}$ has the property $\mathcal{E}$. In this case a winning strategy for $\Omega$-player in the game $G_{(\mathcal{P},\mathcal{P},\mathcal{L})}(X)$ is winning in in the game $G_{(\mathcal{E},\mathcal{F},\mathcal{L})}(X)$ and a winning strategy for $\Sigma$-player in the game $G_{(\mathcal{E},\mathcal{F},\mathcal{L})}(X)$ is winning in the game $G_{(\mathcal{P},\mathcal{P},\mathcal{L})}(X)$.

Definition 11.3. We say that a sequence $\{H_n : n \in \mathbb{N}\}$ of subsets of a space $X$ has the property $M$ if $\cap\{cl_XH_n : n \in \mathbb{N}\} \neq \emptyset$.

Let $exp(X)$ be the family of all subsets of the space $X$.

The corresponding game $G_{(\mathcal{P},exp(X))}(X)$ was defined by E.Michael [47] and will be denoted by $G_M(X)$ or $G(X)$.

For the formulation of some other examples we need some auxiliary notions.

If $\omega$ is a strategy for $\Omega$-player in the game $G_{(\mathcal{P},\mathcal{F},\mathcal{L})}(X)$ and each $\omega$-play has the property $\mathcal{P}$, then we say that $\omega$ is an $\mathcal{P}$-strategy.

Our next example is the following property.

Definition 11.4. We say that a sequence $\{H_n : n \in \mathbb{N}\}$ of open nonempty subsets of a space $X$ has the property $k$ if $K = \cap\{cl_XH_n : n \in \mathbb{N}\}$ is a non-empty compact subset of $X$ and, for each open set $U \supset K$ there is some $n \in \mathbb{N}$ such that $H_m \subset U$ for $m \geq n$.

The game corresponding to this property will be denoted by $G_{(k,\mathcal{F},\mathcal{L})}(X)$.

If $\omega$ is a strategy for $\Omega$-player in the game $G_{(\mathcal{P},\mathcal{F},\mathcal{L})}(X)$ and each $\omega$-play has the property $k$, then we say that $\omega$ is a $k$-strategy. Any $k$-strategy of the player $\Omega$ is a winning strategy for the player $\Omega$ in the game $G_{(k,\mathcal{F},\mathcal{L})}(X)$.

Our last example is the following property.

Definition 11.5. We say that a sequence $\{H_n : n \in \mathbb{N}\}$ of open nonempty subsets of a space $X$ has the property $\theta$ if $K = \cap\{cl_XH_n : n \in \mathbb{N}\}$ is a singleton subset of $X$ and for each open set $U \supset K$ there is some $n \in \mathbb{N}$ such that $H_m \subset U$ for $m \geq n$.

The game corresponding to this property will be denoted by $G_{(\theta,\mathcal{F},\mathcal{L})}(X)$.

By definition, the property $\theta$ is stronger than the property $k$.

In a compact space $X$ every monotone sequence of non-empty sets has the property $k$.

We say that a mapping $\varphi : X \rightarrow Y$ is dense-open if the set $\varphi(X)$ is dense in $Y$ and for any non-empty open subset $U$ of $X$ there exists an open non-empty subset $V$ of $Y$ such that $V \subset cl_Y(V \cap \varphi(U))$.

Any open mapping is dense-open. Any closed continuous irreducible mapping is dense-open. A mapping $\varphi : X \rightarrow Y$ is irreducible if $Y = \varphi(X) \neq \varphi(A)$ for any proper closed subset $A$ of the space $X$.

The technique of open mappings is very important in the theory of topological games.
**Theorem 11.6.** Let $X \varphi : X \rightarrow Y$ be a dense-open continuous mapping of a space $X$ into a space $Y$. For any strategy $\omega$ of the player $\Omega$ in the Banach-Mazur game $G_{BM}(X)$ on $X$ there exists a strategy $s$ of the player $\Omega$ in the Banach-Mazur game $G_{BM}(Y)$ on $Y$ such that:

1. If the $\omega$ is a winning strategy, then $s$ is a winning strategy too.

2. If $\mathcal{P}$ is a property of open subsets which is preserved by the continuous mappings for any $\omega$-play $\{A_n, B_n : n \in \mathbb{N}\}$ the sequence $\{A_n, B_n : n \in \mathbb{N}\}$ has the property $\mathcal{P}$, then for any $s$-play $\{C_n, D_n : n \in \mathbb{N}\}$ the sequence $\{A_n, B_n : n \in \mathbb{N}\}$ has the property $\mathcal{P}$ too.

**Proof.** We can assume that for any $\omega$-play $\{A_n, B_n : n \in \mathbb{N}\}$ we have $cl_X B_n \subset A_n$ for every $n \in \mathbb{N}$.

For each non-empty open subset $U$ of $X$ we fix two non-empty open subsets $l(U)$ and $t(U)$ of $Y$ such that $cl_Y t(U) \subset l(U) \subset cl_Y (l(U) \cap \varphi(U))$.

If $C_1$ is an open non-empty subset of the space $Y$, then we put $r_1(C_1) = \varphi^{-1}(C_1)$, $k_1(C_1) = \omega(r_1(C_1))$ and $D_1 = s(k_1(C_1))$. Assume that $n \in \mathbb{N}$ and $r_i(C_i) \subset X$, $k_i(C_i) \subset X$, $D_i = s(C_i, D_1, ..., C_i) \subset Y$ are constructed for each $i \leq n$. Fix an open non-empty subset $C_{n+1}$ of $Y$. We put $r_{n+1}(C_{n+1}) = \omega(r_1(C_1), k_1(C_1), ..., r_n(C_n), k_n(C_n), r_{n+1}(C_{n+1}))$ and $D_{n+1} = s(C_1, D_1, ..., C_n, D_n, C_{n+1}) = t(k_{n+1}(C_{n+1}))$. The function $s$ is the desired strategy on $Y$.

**Corollary 11.7.** Assume that a space $X$ contains some dense complete $A(k)$-subspace. Then the player $\Omega$ has a winning strategy $\omega$ in the game $G_{(k, \exp(X))}(X)$.

**Corollary 11.8.** Let $X$ be a space, $f \in LC(X)$, $S$ be a dense subspace of the space $Y_f$, $Z$ be a compactification of the space $X$ and $\mathcal{P}$ be a property of open subsets of $X$ such that any $\mathcal{P}$-subset is a $k$-sequence. The following assertions are equivalent:

1. The set $S_{\mathcal{P}, Z}(f)$ contains a dense $G_\delta$-subset of the space $C(Z|X)$.

2. The space $Y_f$ contains a dense $\check{C}$ech-complete subspace.

3. The player $\Omega$ has a winning strategy in the game $G_{(\mathcal{P}, \check{C}, \check{C}, \check{C})}(X)$.

**Proof.** Implications $1 \rightarrow 2 \rightarrow 1$ follows from Theorem 10.3. For the properties $\{k, \theta\}$ the implications $2 \rightarrow 3 \rightarrow 2$ were proved in [20, 21]. The proof of the general case is similar.

Fix a sequence $\{D_n : n \in \mathbb{N}\}$ of dense open subsets of $C(Z|X)$ such that $D = \cap \{D_n : n \in \mathbb{N}\} \subseteq S_{\mathcal{P}, Z}(f)$. Obviously, the set $D$ is a dense $G_\delta$-subset of $C(Z|X)$. 

\[
\square
\]
Part 3. APPLICATIONS TO PROBLEM OF DIFFERENTIABILITY

12 Derivatives

Fix an open non-empty subset $U$ of a normed space $L$ and a function $f : U \rightarrow \mathbb{R}$.

The directional derivative of $f$ at $x \in U$ in the direction $h \in L$, denoted by the symbol $f'(x; h)$, is defined by the equation $f'(x; h) = \lim_{t \to 0^+} (f(x + th) - f(x)) : t$, whenever the limit on the right exists.

The Gâteaux differential generalizes the idea of a directional derivative. We say $f$ is Gâteaux differentiable at $x \in U$ if there is a bounded and linear functional $A_x \in \text{Fun}_c(L)$ such that $\lim_{t \to 0^+} (f(x + th) - f(x)) : t = A_x(h)$ for every $h \in L$. The operator $A_x$ is called the Gâteaux derivative of $f$ at $x$. We have $A_x(h) = f'(x; h)$.

We say $f$ is Fréchet differentiable at $x \in U$ if there is a continuous linear functional $A_x \in \text{Fun}_c(L)$ such that $\lim_{v \to 0} (f(x + v) - f(x)) : \|v\| = A_x(h)$ for every $h \in L$. The operator $A_x$ is called the Fréchet derivative of $f$ at $x$.

A functional $A$ is called a subgradient at $x_0 \in U$ if $f(x) \geq f(x_0) + A(x - x_0)$. The set of all subgradients at $x_0$ is called the subdifferential at $x_0$ and is denoted $\partial f(x_0)$. The subdifferential is always a convex closed set. It can be an empty set.

The subdifferential on convex functions was introduced by J. J. Moreau and R. T. Rockafellar in the 1960s. The generalized subdifferential for nonconvex functions was introduced by F.H. Clarke and R.T. Rockafellar in the 1980s.

Some generalizations of the above concepts are obtained by the following scheme. On normed (or locally convex linear topological spaces) $L$ are selected the families $D(L)$ of continuous functionals $A : L \rightarrow \mathbb{R}$. If $A_x \in D(L)$ in the above definitions, then we obtain the desired extensions of the Gâteaux derivative, Fréchet derivative and subdifferential.

If $f$ is a convex continuous functional on a domain $D$ of a Banach space $B$, then the set of points of Fréchet differentiability is a $G_δ$-subset of $B$.

In [37], P. G. Larman and R. R. Phelps has aroused the following problem: Let $f$ be a convex continuous functional on a Banach space $B$ for which the set $G$ of points of Gâteaux differentiability is dense in $B$. Is it true that the set $G$ is generic, i.e. contains a dense $G_δ$-subset? The first examples of a convex continuous functional on a Banach space $C(X)$ for which the set $G$ of points of Gâteaux differentiability is dense and non-generic in $C(X)$ were constructed in [7]. In the following section the examples of such kind are constructed too.
13 On Gâteaux differentiability of convex functionals

If $L$ and $L'$ are topological linear spaces, then the operator $\psi : L \rightarrow L'$ is Gâteaux differentiable at $b \in L$ if the limit $d'\psi(b,x) = \lim_{t \to 0} t^{-1}(\psi(b + tx) - \psi(b))$ there exists for each $x \in L$.

Let $X$ be a space $X$ and $f : X \rightarrow \mathbb{R}^\infty$ be a proper bounded from below function (not obligatory lower semi-continuous). The subspace $Y = \text{dom}(f)$ is non-empty.

Fix $\Omega \in \{k, \theta\}$. Denote $X_{\Omega,f} = \{g \in S_\Omega(f) : M_{X,f}(g) \neq \emptyset\}$ and $M_{\Omega,f} = M_{X,f}[X_{\Omega,f}].$

Obviously, $m_f : C(X) \rightarrow \mathbb{R}$, where $m_f(g) = \inf_{x \in C(X)} f(x)$ for each $g \in C(X)$, is a convex functional on $C(X)$. Thus for any $g, h \in C(X)$ there exists the limit $d'm_f(g,h) = \lim_{t \to 0} t^{-1}(m_f(g + th) - m_f(g)).$

For the functional $m_f$ it is true the analog of the Banach “peak theorem”.

**Theorem 13.1.** For the function $g \in C(X)$ the following assertions are equivalent:

(i) The functional $m_f$ is Gâteaux differentiable at $g$.

(ii) Any minimizing sequence $\{x_n \in \text{dom}(f)\}$ for the function $f + g$ has a unique cluster point in $\beta X$, i.e. in $\beta X$ there exists the limit $\lim_{n \to \infty} x_n$.

**Proof.** Fix $\varphi \in LC(X)$. For each $x \in \beta X$ we put $\beta \varphi(x) = \inf \{\inf_{x \in U} \beta \varphi : x \in U, U \text{ is open in } \beta X\}$, if $x \in \text{cl}_X \text{dom}(\varphi)$, and $\beta \varphi(x) + \infty$, if $x \notin \text{cl}_X \text{dom}(\varphi)$. Obviously, $\beta \varphi \in LC(\beta X)$ and $\beta \varphi = \beta \varphi X$. Moreover, any minimizing sequence $\{x_n \in \text{dom}(f)\}$ for the function $\varphi$ is a minimizing sequence for the function $\beta \varphi$ too. Hence there exists $c \in \text{cl}_X \text{dom}(\varphi)$ such that $\beta \varphi(c) = \inf_{x \in C(X)} \beta \varphi = \inf_{x \in C(X)} \beta \varphi(c)$. If $\varphi \in C(X)$, then $\beta \varphi \in C(X)$.

Assume that the minimizing sequence $\{x_n \in \text{dom}(f) : n \in \mathbb{N}\}$ for the function $f + g$ has two distinct cluster points $a, b \in \beta X$. On $\beta X$ there exist two open subsets $U, V$ and a continuous function $\psi : \beta X \rightarrow I$ such that $a \in U \subseteq \psi^{-1}(0)$ and $b \in V \subseteq \psi^{-1}(1)$. If $h = \psi|X$, then $\lim_{t \to 0} t^{-1}(m_f(g + th) - m_f(g)) = 0$ and $\lim_{t \to 0} t^{-1}(m_f(g + th) - m_f(g)) = 1$. The implication (i) $\rightarrow$ (ii) is proved.

Suppose now that any minimizing sequence $\{x_n \in \text{dom}(f)\}$ for the function $f + g$ has a unique cluster point in $\beta X$. There exists a unique point $c \in \beta X$ such that $c = \lim_{n \to \infty} x_n$ for any minimizing sequence $\{x_n \in \text{dom}(f)\}$ for the function $f + g$. Then $\inf_{x \in C(X)} (f + g) = \inf_{x \in C(X)} f + g = \inf_{x \in C(X)} f + g$. Fix $h \in C(X)$. In this case we have $\lim_{t \to 0} t^{-1}(m_f(g + th) - m_f(g)) = \beta h(c)$. The implication (ii) $\rightarrow$ (i) is proved.

The proof of the following theorem is similar.

**Theorem 13.2.** For the function $g \in C(X)$ the following assertions are equivalent:

(i) The functional $m_f$ is Fréchet differentiable at $g$.

(ii) Any minimizing sequence $\{x_n \in \text{dom}(f)\}$ for the function $f + g$ is almost constant, i.e. there exists $m \in \mathbb{N}$ such that $x_n = x_m$ for any $n \geq m$.
(iii) There exist a point $x_0$ and a number $\epsilon = \epsilon(f,g)$ such that $f(x) + g(x) \geq f(x_0) + g(x_0) + \epsilon$ for each $x \in X \setminus \{x_0\}$.

**Corollary 13.3.** The points of Fréchet differentiability of the functional $m_f$ is is open in the space $C(Z|X)$.

**Remark.** If $g \in S_0(f)$, then the functional $m_f$ is Gâteaux differentiable at $g$.

**Example 13.4.** Let $Y$ be the subspace of rational numbers of the space $I$ and $X = Z = \beta Y$. As in Example 10.5, consider that $Y = \{r_n : n \in \mathbb{N}\}$, $f(r_n) = n$ and $f(x) = +\infty$ for all $n \in \mathbb{N}$ and $x \in X \setminus Y$. Then:
1. The set $S(\theta,Z)$ is a dense open subset of $C(Z|X)$.
2. The set $C(Z|X) \setminus S(\theta,Z)$ is a dense $G_\delta$-subset of $C(Z|X)$.
3. The set $S(\theta,Z)$ is the set of points of Fréchet differentiability of the functional $m_f$.
4. The set of points of Fréchet differentiability of the functional $m_f$ is empty.

**Example 13.6.** Let $X = \{(x,0) : x \in I\} \cup \{(n^{-1},m^{-1}) : n \leq m, n,m \in \mathbb{N}\}$ be a subspace of Euclidean plane $\mathbb{E}^2$.

Consider that $f$ is a continuous function on $X$. Then:
1. The set $S(\theta,Z)$ is a dense $G_\delta$-subset of $C(Z|X)$.
2. The set $S(\theta,Z)$ is the set of points of Gâteaux differentiability of the functional $m_f$.
4. The set $F$ of points of Fréchet differentiability of the functional $m_f$ is open and dense in $C(Z|X)$ and $F \neq S(\theta,Z)$.

**Example 13.7.** Let $B$ be a normed space and $K$ be a closed convex non-empty subset of $B$. Denote by $\text{ind}_K(x)$ the indicator function of $K$: $\text{ind}_K(x) = 0$ if $x \in K$ and $\text{ind}_K(x) = +\infty$ if $x \in B \setminus K$. Then any point $b \in B$ determine on $B$ the non-negative lower semi-continuous function $I_{(K,b)}(x) = \|x - b\| + \text{ind}_K(x)$.

We have that $\text{dom}(I_{(K,b)}) = K$ and the function $I_{(K,b)}$ is continuous on $K$. The minimization problem $(B, I_{(K,b)})$ was studied by many authors (see [29]).

**Example 13.8.** (A. Królik [42]). A subset $E$ of the closed interval $I$ is called a $\lambda$-set if every countable subset of $E$ is a $G_\delta$-subset of the space $E$ (see [43]). Fix an uncountable $\lambda$-set of $I$. We put $X = \# I \setminus E$.

D. H. Fremlin in [33] has proved that the following assertion is consistent with ZFC-axioms:

A1. $X$ is analytic set, i.e. $X$ is a continuous image of the space of irrational numbers.
In the virtue of the Fremlin’s result, in [42] A. Królak has proved that the following assertions are consistent with ZFC-axioms:

A2. The space $X$ is not completely metrizable.
A3. The player $\Omega$ has no winning strategy in the Michael’s game $G_M(X)$.
A4. For any function $f \in LC(X)$ the set $S_{(M,X)}(f)$ is a residual (generic) set of the space $C(X)$.

From the Assertion A4 it follows:
A5. For any function $f \in LC(X)$ the set $S_{(\theta,X)}(f)$ is a residual set of the space $C(X)$.
A6. The set $S_{(\theta,Z)}(f)$ is the set of points of Gâteaux differentiability of the functional $m_f$ for any function $f \in LC(X)$.
A7. Any closed subspace of the space $X$ contains a dense completely metrizable subspace.
A8. If $f \in LC(X)$, then the subspace $Y_f = \text{dom}(f)$ in the topology $\mathcal{L}_f$ contains a dense completely metrizable subspace.
References


