Optimal recovery of functionals and operators with relation to approximate solution of PDE’s

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In this talk we will speak about:

1. Dnepropetrovsk National University;
2. Operators recovery problem;
3. Optimal recovery of solution to PDEs.
In what follows we would consider linear normed spaces:

1. $C(\Omega)$ - space of continuous on $\Omega$ functions:
   \[
   \|x\|_{C(\Omega)} = \max\{|x(t)| : t \in \Omega\};
   \]

2. $L_\infty(\Omega)$ - space of essentially bounded on $\Omega$ functions:
   \[
   \|x\|_{L_\infty(\Omega)} = \text{esssup}\{|x(t)| : t \in \Omega\};
   \]

3. $L_p(\Omega), 1 \leq p < \infty$, - space of $p$-integrable on $\Omega$ functions:
   \[
   \|x\|_{L_p(\Omega)} = \left(\int_{\Omega} |x(t)|^p \, dt \right)^{1/p}.
   \]
Operators on linear spaces

Let $X$ and $Y$ be linear normed spaces on $\mathbb{R}$. Consider linear operator $A : X \to Y$.

Examples of functionals:

1. Integration
   $Ax = \int_{\Omega} x(t) \, dt$;

2. Integration with weight
   $Ax = \int_{\Omega} w(t)x(t) \, dt$;

3. Differentiation at fixed point
   $Ax = x'(t_0)$. 
Examples of operators:

1. Differentiation

\[ Ax = x'; \]

2. Convolution

\[ Ax(t) = \int_\Omega K(t, u)x(u)\, du, \quad t \in \Omega; \]

3. Solution of ODE or PDE, e.g. consider Dirichlet’s problem for Laplace’s equation

\[
\begin{cases}
\Delta u = 0, & \text{on } \Omega, \\
u = \varphi, & \text{on } \partial \Omega.
\end{cases}
\]

For \( \varphi \in C(\partial \Omega) \), it has a unique solution \( u \in C^2(\Omega) \cap C(\overline{\Omega}) \) that can be considered as the linear operator \( A : \varphi \rightarrow u \).
There are many good and important operators.
How to compute them?
How to compute them in the best possible way?
Why such questions are important?

1. **computational reason**: integration and differentiation are the limits, but how one could compute them with limited number of operations?

2. **definition reason**: if output of operator is a function when we need to present this function in terms of "simple" functions?

3. **practical reason**: during experiment we can perform only finite number of measurements of process under study.
Let $X$ and $Y$ be linear normed spaces and let $A : X \rightarrow Y$ be linear operator.

1. Compute $Ax$ for every $x \in X$ (or in some meaningful subset of $X$);

2. We know some finite information $I(x)$ about function $x$. Information is an operator $I : X \rightarrow \mathbb{R}^N$, $N \in \mathbb{N}$, presented in the form $I(x) = (\varphi_1(x), \varphi_2(x), \ldots, \varphi_N(x))$ where $\varphi_k : X \rightarrow \mathbb{R}$ are the linear functionals;

3. Find generic formula, i.e. operator (method of recovery) $\Phi : \mathbb{R}^N \rightarrow Y$ that would compute $Ax$ for every $x \in X$ (or in some meaningful subset of $X$).
Examples

Quadrature (cubature) formulas

$$
\int_{\Omega} x(t) \, dt \approx \frac{|\Omega|}{N} x(t_1) + \frac{|\Omega|}{N} x(t_2) + \ldots + \frac{|\Omega|}{N} x(t_N),
$$

where $|\Omega|$ stands for the volume of $\Omega$ and points $t_1, t_2, \ldots, t_N$ are fixed.

More generally,

$$
\int_{\Omega} w(t) x(t) \, dt \approx \alpha_1 x(t_1) + \alpha_2 x(t_2) + \ldots + \alpha_N x(t_N),
$$

where points $t_1, t_2, \ldots, t_N$ and coefficients $\alpha_1, \alpha_2, \ldots, \alpha_N$ are fixed.

Finite differences:

$$
x'(t_0) \approx \frac{x(t_0 + h_0) - x(t_0 - h_0)}{2h_0},
$$

where number $h_0$ is fixed.
Naturally, the question of the best recovery of operator $A$ is interesting. But what should we understood by "optimal"?

\[
\inf_{\Phi} \sup_{x \in X} \|Ax - \Phi(I(x))\|_Y = \infty, \]

if $\Phi(I) \not\equiv A$.

Reduce set in sup as follows. Consider bounded subset $\mathcal{M} \subset X$ and define

\[
E(A, \mathcal{M}, I) = \inf_{\Phi} \sup_{x \in \mathcal{M}} \|Ax - \Phi(I(x))\|_Y. 
\]

*Error of optimal recovery of operator $A$ on class $\mathcal{M}$ given information $I$.\*
Example: upper bound

Consider $X = L_\infty([0, 1])$,

$$Ax = \int_0^1 x(t) \, dt, \quad I(x) = x(t_0).$$

Also for $K > 0$, let

$$M = H^1_K := \{ x \in L_\infty([0, 1]) : |x'(t)| \leq K \} .$$

Then,

$$E(A, H^1_K, I) \leq \sup_{x \in H^1_K} \left| \int_0^1 x(t) \, dt - x(t_0) \right| = \sup_{x \in H^1_K} \left| \int_0^1 [x(t) - x(t_0)] \, dt \right|$$

$$\leq \sup_{x \in H^1_K} \int_0^1 |x(t) - x(t_0)| \, dt = \sup_{x \in H^1_K} \int_0^1 \left| \int_{t_0}^t x'(u) \, du \right| \, dt$$

$$\leq K \int_0^1 |t - t_0| \, dt = K \left[ \frac{t_0^2 + (1 - t_0)^2}{2} \right].$$
Let $X$ be a Banach space and $X^*$ be its dual.

**Theorem**

Let $A, \varphi_1, \ldots, \varphi_N \in X^*$ be given. Then

$$\inf_{\lambda_1, \ldots, \lambda_N} \left\| A - \sum_{k=1}^{N} \lambda_k \varphi_k \right\|_{X^*} = \sup_{\|x\|_X \leq 1} \left\{ A(x) : \varphi_1(x) = 0, \ldots, \varphi_N(x) = 0 \right\}.$$
Proof of Nikolskii duality theorem

Would be presented on the blackboard.
Let $X$ be a Banach space and $X^*$ be its dual.

**Theorem**

Let $A, \varphi_1, \ldots, \varphi_N \in X^*$ be given. Then

$$
\inf_{\lambda_1, \ldots, \lambda_N} \sup_{\|x\|_X \leq 1} \left| A(x) - \sum_{k=1}^{N} \lambda_k \varphi_k(x) \right| = \sup_{\|x\|_X \leq 1} A(x),
$$

where $\varphi_1(x) = \ldots = \varphi_N(x) = 0$.

- **Milestone in Approximation Theory**
- **Powerful tool to solve various Extremal problems**

S.M. Nikolskii (1950): Optimal quadrature formulas
Let $\mathcal{M} \subset X$ be centrally symmetric convex body. Similarly it is easy to prove

**Theorem**

Let $A, \varphi_1, \ldots, \varphi_N \in X^*$ be given. Then

$$\inf_{\lambda_1, \ldots, \lambda_N} \sup_{x \in \mathcal{M}} \left| A(x) - \sum_{k=1}^{N} \lambda_k \varphi_k(x) \right| = \sup_{x \in \mathcal{M}} A(x) \quad \varphi_1(x) = \ldots = \varphi_N(x) = 0$$

This is the duality theorem for optimal recovery of bounded functional $A$ on the class $\mathcal{M}$ with the help of linear methods that use information about $x \in \mathcal{M}$ given by functionals $\varphi_1, \ldots, \varphi_N$. 
If $\Phi : \mathbb{R}^N \to \mathbb{R}$ is an arbitrary method of recovery of functional $f$, then

$$\sup_{x \in \mathcal{M}} |A(x) - \Phi(I(x))| \geq \sup_{\varphi_1(x) = 0, \ldots, \varphi_N(x) = 0} |A(x) - \Phi(0, \ldots, 0)|$$

$$\geq \sup_{\varphi_1(x) = 0, \ldots, \varphi_N(x) = 0} \max \{|A(x) - \Phi(0, \ldots, 0)|, |A(x) + \Phi(0, \ldots, 0)|\}$$

$$\geq \sup_{x \in \mathcal{M}} |A(x)|.$$  

In view of previous theorem, from this estimation, we obtain that among all optimal methods of recovery there exists a linear method.
For linear operator $A : X \to Y$ and symmetric convex body $\mathcal{M} \subset X$ we can follow Smolyak’s idea to prove

$$E(A, \mathcal{M}, I) \geq \sup_{x \in \mathcal{M} : I(x) = 0} \|Ax\|_Y.$$

Example:

$$E \left( \int_0^1 x(t) \, dt, H^1_K, x(t_0) \right) \geq \sup_{x \in H^1_K : x(t_0) = 0} \left| \int_0^1 x(t) \, dt \right|$$

$$\geq \int_0^1 |t - t_0| \, dt = K \frac{t_0^2 + (1 - t_0)^2}{2}.$$

Compare with the upper estimate

$$E \left( \int_0^1 x(t) \, dt, H^1_K, x(t_0) \right) \leq K \frac{t_0^2 + (1 - t_0)^2}{2}.$$
Return to one-point quadrature on $H^1_K$

\[
E \left( \int_0^1 x(t) \, dt, H^1_K, x(t_0) \right) = \frac{K \left[ t_0^2 + (1 - t_0)^2 \right]}{2} \geq \frac{K}{4}
\]

\[
= E \left( \int_0^1 x(t) \, dt, H^1_K, x \left( \frac{1}{2} \right) \right).
\]

So, some information might be better than other information.

Let $\mathcal{I}$ be the type of information (subset of operators from $X$ to $\mathbb{R}^N$), e.g. values at $N$ points, averages over $N$ small measurement intervals, first $N$ Fourier coefficient, $N$ coefficients of wavelet expansion, etc. Set

\[
E (A, \mathcal{M}, \mathcal{I}) := \inf_{I \in \mathcal{I}} E (A, \mathcal{M}, I).
\]

**Error of optimal recovery of operator $A$ on class $\mathcal{M}$ given information of type $\mathcal{I}$.**
Optimal quadrature formula on $H^1_K$

If $I_N$ is values of function $x \in X$ at $N$ points then

$$E \left( \int_0^1 x(t) \, dt, H^1_K, I_1 \right) = \frac{K}{4}.$$ 

In general case

$$E \left( \int_0^1 x(t) \, dt, H^1_K, I_N \right) = \frac{K}{4N^2},$$

optimal information is

$$I(x) = \left( x\left(\frac{1}{2N}\right), x\left(\frac{3}{2N}\right), \ldots, x\left(\frac{2N-1}{2N}\right) \right),$$

optimal method of recovery (optimal quadrature formula) is

$$\Phi(I(x)) = \frac{1}{N} \left[ x\left(\frac{1}{2N}\right) + x\left(\frac{3}{2N}\right) + \ldots + x\left(\frac{2N-1}{2N}\right) \right].$$
Let $X = H$ be the Hilbert space, $A : H \rightarrow H$ be the bounded linear operator and $B_H$ be the unit ball in $H$. Let also $\varphi_1, \varphi_2, \ldots, \varphi_N \in H$ and $A^*$ be the conjugated operator to $A$, i.e. $\langle Ax, y \rangle = \langle x, A^*y \rangle$.

**Theorem**

If $I = (A^*\varphi_1, A^*\varphi_2, \ldots, A^*\varphi_N)$ then

$$E(A, B_H, I) = \sup_{x \in B_H} \parallel Ax \parallel_H.$$

Proof would be presented on the blackboard.
Results would be presented on the blackboard.