SMOOTHNESS BEYOND DIFFERENTIABILITY

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Abstract. The traditional concept of smoothness concerns differentiability, and we consider a function to be smooth if it is \( k \)-times differentiable. This concept is important in many aspects of numerical analysis, and the quality of reconstructions of signals from certain given observations depend on this information. For instance, a \( k \)-times differentiable function on the unit cube \([0,1]^d\) can be approximated by certain splines at the order \( n^{-k/d} \), where \( n \) denotes the number of information (values at a grid).

In inverse problems, when we are given data
\[
y = Tx + \delta \xi,
\]
where
\begin{itemize}
  \item \( T: X \to Y \) is a bounded (compact) linear operator in Hilbert space,
  \item \( \xi \) is (unknown) noise with \( \|\xi\|_Y \leq 1 \), and
  \item \( \delta > 0 \) is the noise level,
\end{itemize}
and the goal is to approximate the unknown \( x \in X \) then the quality of the reconstruction for \( x \) depends on properties of the operator \( T \). This was observed very early in the analysis of inverse problems, as e.g. in [1], where it is assumed that \( x \in \mathcal{R}(T^*) \), with \( T^*: Y \to X \) being the adjoint operator. This was later extended to cases \( x \in \mathcal{R}((T^*T)^p) \) for some power \( p > 0 \), and the seminal monograph is [3] from Tartu!

The questions discussed in this talk will thus be: Are these concepts related? If yes, is there a calculus for smooth elements in Hilbert space, similar to the one for differentiable functions? In a nutshell, the traditional smoothness can be regarded as \( f \in \mathcal{D}(D) \), i.e., the function is differentiable, if it belongs to the domain of some (unbounded) differential operator, and thus in the range of the inverse operator \( D^{-1} \). Often this is compact, and then we are exactly in the situation as in inverse problems.

We shall highlight these similarities in more detail, and we shall outline some results from interpolation theory for smooth elements in Hilbert space, as this was established in a series of papers, specifically in [2].

References