On the osculator Lorentz spheres of timelike parallel $p_i$-equidistant ruled surfaces in the Minkowski 3-space $R^3_1$

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Abstract. In this paper, we present radii and curvature axes of osculator Lorentz spheres of the timelike parallel $p_i$-equidistant ruled surfaces with a timelike base curve in the Minkowski 3-space $R^3_1$ and give the arc lengths of indicatrix curves of timelike base curves of these surfaces.

1. Introduction

I. E. Valeontis [3] defined parallel $p$-equidistant ruled surfaces in $E^3$ and gave some results related to the striction curves of these surfaces.

M. Masal and N. Kuruoğlu [2] studied arc lengths, curvature radii, curvature axes, spherical involute and areas of real closed spherical indicatrix curves of base curves of parallel $p$-equidistant ruled surfaces in $E^3$. And also, M. Masal and N. Kuruoğlu [1] defined timelike parallel $p_i$-equidistant ruled surfaces with a timelike base curve in the Minkowski 3-space and have studied dralls, the shape operators, Gaussian curvatures, mean curvatures, shape tensor, $q^{th}$ fundamental forms of these surfaces.

This paper is organized as follows. In Section 3 we have found radii and curvature axes of osculator Lorentz spheres of the timelike parallel $p_i$-equidistant ruled surfaces with a timelike base curve in the Minkowski 3-space. And later in Section 4 we have given arc lengths of indicatrix curves of these surfaces.

2. Preliminaries

Let $\alpha : I \rightarrow R^3_1$, $\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$ be a differentiable unit speed timelike curve in the Minkowski 3-space, where $I$ is an open interval.

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in $\mathbb{R}$ containing the origin. Let $V_1$ be the tangent vector field of $\alpha$, $D$ be the Levi-Civita connection on $R^3$, and $D_{V_1}V_1$ be a spacelike vector. If $V_1$ moves along $\alpha$, then a timelike ruled surface $M$ which is given by the parametrization

$$\varphi(t, v) = \alpha(t) + vV_1(t)$$

is obtained. Let $\{V_1, V_2, V_3\}$ be an orthonormal frame field along $\alpha$ in $R^3$, where $V_2$ and $V_3$ are spacelike vectors. If $k_1$ and $k_2$ are the natural curvature and torsion of $\alpha(t)$, respectively, then for $\alpha$ the Frenet formulas are given by (see [4])

$$V_1' = k_1V_2, \quad V_2' = k_1V_1 - k_2V_3, \quad V_3' = k_2V_2.$$ (2.1)

Using $V_1 = \alpha'$ and $V_2 = \frac{\alpha''}{\|\alpha''\|}$, we have $k_1 = \|\alpha''\| > 0$, where $\|\alpha''\|$ means derivative with respect to time $t$ (see [1]).

**Definition 2.1** ([1]). The planes corresponding to subspaces $Sp\{V_1, V_2\}$, $Sp\{V_2, V_3\}$ and $Sp\{V_3, V_1\}$ along striction curves of timelike ruled surface $M$ are called **asymptotic plane, polar plane and central plane**, respectively.

Let us suppose that $\alpha^* = \alpha^*(t^*)$ is another differentiable timelike curve with arc-length and $\{V_1^*, V_2^*, V_3^*\}$ is the Frenet frame of this curve in three dimensional Minkowski space $R^3$. Hence, we define timelike ruled surface $M^*$ parametrically as follows:

$$\varphi^*(t^*, v^*) = \alpha^*(t^*) + v^*V_1^*(t^*), \quad (t^*, v^*) \in I \times \mathbb{R}.$$ 

**Definition 2.2** ([1]). Let $M$ and $M^*$ be two timelike ruled surfaces and let $p_1$, $p_2$ and $p_3$ be the distances between the polar planes, central planes and asymptotic planes, respectively. If the directions of $M$ and $M^*$ are parallel and the distances $p_i$, $1 \leq i \leq 3$, of $M$ and $M^*$ are constant, then the pair of ruled surfaces $M$ and $M^*$ is called **timelike parallel** $p_i$-equidistant ruled surfaces with a timelike base curve. If specifically $p_i = 0$, then this pair of ruled surfaces is named as **timelike parallel** $p_i$-equivalent ruled surfaces with a timelike base curve, where the base curves of ruled surfaces $M$ and $M^*$ are of class $C^2$.

Therefore the pair of timelike parallel $p_i$-equidistant ruled surfaces are defined parametrically as

$$M : \varphi(t, v) = \alpha(t) + vV_1(t), \quad (t, v) \in I \times \mathbb{R},$$

$$M^* : \varphi^*(t^*, v^*) = \alpha^*(t^*) + v^*V_1^*(t^*), \quad (t^*, v^*) \in I \times \mathbb{R},$$

where $t$ and $t^*$ are the arc parameters of curves $\alpha$ and $\alpha^*$, respectively. Let the striction curve of $M$ be the base curve of $M$ and let $\alpha^*$ be a base curve.
of $M^*$. In this case we can write
\[ \alpha^* = \alpha + p_1 V_1 + p_2 V_2 + p_3 V_3, \]
where $p_1(t)$, $p_2(t)$ and $p_3(t)$ are of class $C^2$ (see [1]).

**Theorem 2.1** (see [1], Theorem 3.2 and Corollary 3.1). Let $M$ and $M^*$ be timelike parallel $p_i$-equidistant ruled surfaces.

i) The Frenet vectors of timelike parallel $p_i$-equidistant ruled surfaces $M$ and $M^*$ at $\alpha(t)$ and $\alpha^*(t^*)$ points are equivalent for $\frac{dt^*}{dt} > 0$.

ii) There is a relation between the natural curvatures $k_1(t)$ and $k_1^*(t^*)$ of base curves and the torsions $k_2(t)$ and $k_2^*(t^*)$ of $M$ and $M^*$ as follows:
\[ k_i^* = k_i \frac{dt^*}{dt}, \quad 1 \leq i \leq 2. \]

3. On the osculator Lorentz spheres of timelike parallel $p_i$-equidistant ruled surfaces with a timelike base curve

In this section, we will investigate radii and curvature axes of osculator Lorentz spheres of timelike parallel $p_i$-equidistant ruled surfaces $M$ and $M^*$ with a timelike base curve.

We compute the locus of center of the osculator sphere $S^2_1$ which is the fourth order contact with the base curve $\alpha$ of $M$. Let us consider the function $f$ defined by
\[ f : I \to \mathbb{R} \]
\[ t \to f(t) = \langle \alpha(t) - a, \alpha(t) - a \rangle - R^2, \]
where $a$ and $R$ are the center and radius of $S^2_1$, respectively. Since $S^2_1$ is the fourth order contact with the curve $\alpha$, we can write
\[ f(t) = f'(t) = f''(t) = f'''(t) = 0. \]

From $f(t) = 0$ we have
\[ \langle \alpha(t) - a, \alpha(t) - a \rangle = R^2, \quad (3.1) \]
from $f'(t) = 0$ and $V_1(t) = \alpha'(t)$ we get
\[ \langle V_1(t), \alpha(t) - a \rangle = 0, \quad (3.2) \]
from $f''(t) = 0$ and equation (2.1) we have
\[ \langle V_2(t), \alpha(t) - a \rangle = \frac{1}{k_1(t)}, \quad (3.3) \]
Furthermore, for the vector $\alpha(t) - a$, we can write
\[ \alpha(t) - a = m_1(t)V_1(t) + m_2(t)V_2(t) + m_3(t)V_3(t), \quad m_i(t) \in \mathbb{R}, \quad (3.4) \]
where \( \{V_1, V_2, V_3\} \) is the orthonormal frame field of \( M \). From here we have

\[
\langle \alpha(t) - a, V_1(t) \rangle = -m_1(t), \quad \langle \alpha(t) - a, V_2(t) \rangle = m_2(t), \quad \langle \alpha(t) - a, V_3(t) \rangle = m_3(t).
\]  

(3.5)

From equations (3.2) and (3.3) we get

\[
m_1(t) = 0, \quad m_2(t) = \frac{1}{k_1(t)}.
\]  

(3.6)

From equations (3.1), (3.4) and (3.6) we obtain

\[
R = \sqrt{m_2^2 + m_3^2}
\]  

(3.7)

or

\[
m_3 = \pm \sqrt{R^2 - m_2^2}.
\]  

(3.8)

Using (3.4), for the center \( a \) of \( S_1^2 \), we can write

\[
a = \alpha(t) - \frac{1}{k_1} V_2 - \lambda V_3, \quad \lambda = m_3(t) \in \mathbb{R}.
\]

From \( f''''(t) = 0 \) we have

\[
k_1' \langle V_2(t), \alpha(t) - a \rangle + k_1 \langle V_2'(t), \alpha(t) - a \rangle + k_1 \langle V_2(t), V_1(t) \rangle = 0.
\]

So, from (2.1), (3.5) and (3.6) we obtain

\[
m_3 = \frac{-k_1'}{k_1^2 k_2} = \frac{m_2'}{k_2}.
\]  

(3.9)

Similarly, we compute the locus of center of osculator sphere \( S_1^{*2} \) which is the fourth order contact with the timelike base curve \( \alpha^* \) of \( M^* \). Let us consider the function \( f^* \) defined as

\[
f^* : I \rightarrow \mathbb{R}
\]

\[
t^* \rightarrow f^* (t^*) = \langle \alpha^*(t^*) - a^*, \alpha^*(t^*) - a^* \rangle - R^*^2,
\]

where \( a^* \) and \( R^* \) are the center and the radius of \( S_1^{*2} \). In addition, for the vector \( \alpha^*(t^*) - a^* \), we can write

\[
\alpha^*(t^*) - a^* = m_1(t^*) V_1(t^*) + m_2(t^*) V_2(t^*) + m_3(t^*) V_3(t^*), \quad m_i(t^*) \in \mathbb{R},
\]

where \( \{V_1^*, V_2^*, V_3^*\} \) is the orthonormal frame field of \( M^* \).

In a similar way \( m_1^*(t^*), m_2^*(t^*), m_3^*(t^*), R^* \) and \( a^* \) of \( S_1^{*2} \) are found to be

\[
m_1^*(t^*) = 0, \quad m_2^*(t^*) = \frac{1}{k_1'(t^*)}, \quad m_3^*(t^*) = \frac{m_2'}{k_2'(t^*)}, \quad R^* = \sqrt{m_2'^2 + m_3'^2}
\]  

(3.10)

and

\[
a^* = \alpha^*(t^*) - \frac{1}{k_1} V_2^* - \lambda^* V_3^*, \quad \lambda^* = m_3^*(t^*) \in \mathbb{R}.
\]
Now, we can compute the relations between the radii of osculator Lorentz spheres and curvature axes of the base curves of $M$ and $M^*$. From Theorem 2.1 ii), equations (3.6) and (3.10), we have

$$m_1^*(t^*) = m_1(t) = 0, \quad m_2^*(t^*) = \frac{dt^*}{dt} m_2(t). \quad (3.11)$$

If $\frac{dt}{dt^*}$ is constant, then from Theorem 2.1 ii) we obtain

$$k_1^{*'} = k_1' \left( \frac{dt}{dt^*} \right)^2. \quad (3.12)$$

Hence, using (3.10), (3.12), (3.9) and Theorem 2.1 ii), we find

$$m_3^* = \frac{dt^*}{dt} m_3. \quad (3.13)$$

Combining (3.11), (3.13) and Theorem 2.1 i), we get

$$\alpha^* - a^* = \frac{dt^*}{dt} (\alpha - a).$$

Similarly, combining (3.7), (3.8), (3.11) and (3.13), we have

$$R^{*2} = \left( \frac{dt^*}{dt} \right)^2 R^2$$

or

$$R^* = \left| \frac{dt^*}{dt} \right| R.$$

So, we have proved the following theorem.

**Theorem 3.1.** Let $M$ and $M^*$ be the timelike parallel $p_t$-equidistant ruled surfaces with a timelike base curve.

i) If $q_\alpha$ and $q_{\alpha^*}$ are the curvature axes (the locus of center of osculator Lorentz spheres) of the base curves $\alpha$ and $\alpha^*$ of $M$ and $M^*$, then we have

$$q_{\alpha^*} - \alpha^* = \frac{dt^*}{dt} (q_\alpha - \alpha).$$

ii) If $R$ and $R^*$ are the radiuses of osculator Lorentz spheres of base curves $\alpha$ and $\alpha^*$ of $M$ and $M^*$, then we have

$$R^* = \left| \frac{dt^*}{dt} \right| R.$$
4. Arc lengths of indicatrix curves of the timelike parallel $p_i$-equidistant ruled surfaces with a timelike base curve

In this section, we will investigate arc lengths of indicatrix curves of timelike base curves of the timelike parallel $p_i$-equidistant ruled surfaces $M$ and $M^*$ with timelike base curve.

Since $V_2$ and $V_3$ are spacelike vectors, the curves $(V_2)$ and $(V_3)$ generated by the spacelike vectors $V_2$ and $V_3$ on the pseudosphere $S^2_1$ are called the pseudo-spherical indicatrix curves. The curve $(V_1)$ generated by the vector $V_1$ on the pseudohyperbolic space $H^2_1$ is called indicatrix curve. Let $S_{V_i}$ and $S_{V^*_i}$ denote the arc lengths of indicatrix curves $(V_i)$ and $(V^*_i)$ generated by the vector fields $V_i$ and $V^*_i$, respectively. So we can write

$$S_{V_i} = \int \| V'_i \| \, dt$$

and

$$S_{V^*_i} = \int \| V'^*_i \| \, dt^*, \quad 1 \leq i \leq 3.$$

Using the Frenet formulas and Theorem 2.1 ii), we get

$$S_{V^*_i} = \int k_1 \, dt = S_{V_i}, \quad S_{V^*_2} = \int \sqrt{|k^2_2 - k^2_1|} \, dt = S_{V_2}, \quad S_{V^*_3} = \int |k_2| \, dt = S_{V_3}.$$

where $\frac{dt}{dt^*} > 0$.

Similarly, for the arc lengths $S_{\alpha}$ and $S_{\alpha^*}$ of the indicatrix curves $(\alpha)$ and $(\alpha^*)$ generated by the timelike curves $\alpha$ and $\alpha^*$ on the pseudosphere $S^2_1$, respectively, we can write

$$S_{\alpha} = \int \| \alpha' \| \, dt = \int \, dt$$

and

$$S_{\alpha^*} = \int \| \alpha'^* \| \, dt^* = \int \, dt^*.$$

If $\frac{k_1}{k^*_1}$ is constant, then using Theorem 2.1 ii), we obtain

$$S_{\alpha^*} = \frac{k_1}{k^*_1} S_{\alpha}.$$

Thus we have proved the following theorems.

**Theorem 4.1.** If $S_{V_i}$ and $S_{V^*_i}$, $1 \leq i \leq 3$, are the arc lengths of indicatrix curves of Frenet vectors $V_i$ and $V^*_i$ of timelike base curves $\alpha$ and $\alpha^*$ of the timelike parallel $p_i$-equidistant ruled surfaces $M$ and $M^*$, respectively, then we have

$$S_{V^*_i} = S_{V_i}, \quad 1 \leq i \leq 3.$$

**Theorem 4.2.** Let $S_{\alpha}$ and $S_{\alpha^*}$ be the arc lengths of indicatrix curves of timelike base curves $\alpha$ and $\alpha^*$ of the timelike parallel $p_i$-equidistant ruled
surfaces $M$ and $M^*$, respectively. If \( \frac{k_1}{k_1^*} \) is constant, then we have \( S_{\alpha^*} = \frac{k_1}{k_1^*} S_\alpha \).

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References


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