ACTA ET COMMENTATIONES UNIVERSITATIS TARTUENSIS DE MATHEMATICA
Volume 12, 2008

A conjecture of Bertino: from copulas to graph theory

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Abstract. In this article we prove a conjecture of Salvatore Bertino which centers on a study of the function

$$F(C) = \int_0^{1/2} V_C([t, t+1/2]^2)\, dt, \tag{1}$$

where C is a copula. This function, which arises in the study of the index of dissimilarity between two random variables (see Bertino 1977), is the integral of the C-volumes of the family of squares of sidelength 1/2 sliding from the lower left to the upper right along the ascending diagonal of the unit square. The conjecture made in 1999 is that the minimum value of F among the Bertino copulas is 1/8, which is the value of F at the Fréchet lower bound. The conjecture is established by showing that a certain family of shuffles of M is dense in the set of Bertino copulas and then solving a problem in weighted graphs, which is equivalent to the conjecture in the restricted case.

Introduction

We begin with some observations about the function F defined in (1) to frame the conjecture of Bertino. Since $0 \le V_C([t, t+1/2]^2) \le 1/2$ for each copula C and each $t \in [0, 1/2]$, the values of F lie in the interval $[0, 1/4]$. In fact, there exist copulas for which F takes on the values 0 and $1/4$ – in the latter case, the copula $M(u, v) = \min\{u, v\}$ works as $V_M([t, t+1/2]^2) \equiv 1/2$, and in the former case, the copula K with mass spread uniformly on the union of the line segments between (0,1/2) and (1/2,1) and between (1/2,0) and (1,1/2) works as $V_K([t, t+1/2]^2) \equiv 0$. Of particular importance is the copula $W(u, v) = \max\{u + v - 1, 0\}$ which spreads mass uniformly on the descending diagonal of the unit square, so $V_W([t, t+1/2]^2)$ is equal to $2t$ on $[0, 1/4]$ and $1 - 2t$ on $[1/4, 1/2]$, and thus $F(W) = 1/8$. M and W are both Bertino copulas, while K is not. Therefore the maximum value of F among

Received September 18, 2007.
2000 *Mathematics Subject Classification.* Primary 60E05; Secondary 62H05, 05C90.
Key words and phrases. Bertino copula, copula, Dyck path, shuffle, weighted graph.

the Bertino copulas is 1/4 and the Bertino conjecture is that the minimum value of F among the Bertino copulas is the value at W, i.e., 1/8.

Section 1 contains preliminary definitions and results on shuffles of M and Bertino copulas. In Section 2 critical shuffles are defined and their denseness in the space of Bertino copulas is established. In Section 3 critical graphs are defined and the Bertino conjecture is re-formulated as a problem in graph theory. The proof is given in Section 4.

1. Preliminaries

A *copula* is a function $C : I^2 \to I = [0,1]$ that satisfies the boundary conditions $C(t,0) = C(0,t) = 0$ and $C(t,1) = C(1,t) = t$ for all $t \in I$, and the *two-increasing condition*

$$C(u_1, v_1) - C(u_1, v_2) - C(u_2, v_1) + C(u_2, v_2) \geq 0$$

whenever $u_1 < u_2$ and $v_1 < v_2$ in I. The expression on the left is the *C-volume of* $[u_1, u_2] \times [v_1, v_2]$ and is denoted by $V_C([u_1, u_2] \times [v_1, v_2])$. The set of copulas is a metric space under the uniform metric. The *support* of a copula C is the complement of the union of all open rectangles whose closures have C-volume equal to zero. For an introduction to copulas and more information on the topics in this section, see Nelsen 2006.

Proposition 1. *F is a continuous function on the space of copulas.*

Proof. Let $\epsilon > 0$ be given. If C_1 and C_2 are copulas for which $\big|C_1(u,v) - C_2(u,v)\big| \leq \epsilon$ whenever $(u,v) \in I^2$ then $\big|V_{C_1}([t, t+1/2]^2) - V_{C_2}([t, t+1/2]^2)\big| \leq 4\varepsilon$ for all $t \in [0, 1/2]$ and hence

$$|F(C_1) - F(C_2)| \leq \int_0^{1/2} \big|V_{C_1}([t, t+1/2]^2) - V_{C_2}([t, t+1/2]^2)\big| \, dt \leq 2\varepsilon.$$

□

For each copula C, the function $\delta_C : I \to I$ defined by $\delta_C(t) = C(t,t)$ is the *diagonal section of C*. A *diagonal* is any function $\delta : I \to I$ for which $\delta(0) = 0$, $\delta(1) = 1$, $\delta(t) \leq t$ for all $t \in I$, and $0 \leq \delta(t_2) - \delta(t_1) \leq 2(t_2 - t_1)$ whenever $t_1 < t_2$ in I. Every diagonal section of a copula is a diagonal and conversely (see Fredricks and Nelsen 1997). One easily establishes

Proposition 2. *For each diagonal δ, the function $\hat{\delta} : I \to I$ defined by $\hat{\delta}(t) = t - \delta(t)$ satisfies $\hat{\delta}(0) = \hat{\delta}(1) = 0$ and $\left|\hat{\delta}(t_2) - \hat{\delta}(t_1)\right| \leq |t_2 - t_1|$ for all $t_1, t_2 \in I$. Conversely, if a function $\hat{\delta}$ satisfies the preceding conditions, then $\delta(t) = t - \hat{\delta}(t)$ is a diagonal.*

A *shuffle of M* is a copula whose mass is uniformly distributed on the image of the ascending diagonal Δ of I^2 under the following action: partition I^2 into a finite number of closed vertical strips; permute the strips; and, perhaps, flip some of the strips about their vertical axes of symmetry. Thus, a shuffle of M is determined by a partition P of I, a permutation π on $\mathbf{n} = \{1, 2, ..., n\}$ where n is the number of subintervals of P, and a function $\omega : \mathbf{n} \to \{-1, 1\}$ where $w(i)$ is –1 or 1 depending on whether or not the corresponding strip is flipped. The resulting shuffle of M is denoted by $M(P, \pi, \omega)$. A shuffle of M for which $w \equiv -1$ is called a *flipped shuffle of M.* For each positive integer n, let P_n denote the partition of I consisting of all the multiples of $1/n$. A shuffle of M for which $P = P_n$ is called a *regular* (or, more specifically, an *n-regular) shuffle of M.*

Proposition 3. *If $C = M(P_n, \pi, \omega)$ is a shuffle of M with n even, then the function $V_C(t) \equiv V_C([t, t + 1/2]^2)$ is piecewise linear on $[0, 1/2]$ with constant slope on each subinterval of the partition $P_n/2$ of $[0, 1/2]$. If, in addition, $\omega(i) = 1$ whenever i is a fixed point of π or $|\pi(i) - i| = n/2$, then the function V_C has constant slope on each subinterval of the partition Q_n of $[0, 1/2]$ consisting of all multiples of $1/n$.*

Proof. Note that V_C is piecewise linear on $[0, 1/2]$ for any shuffle C of M, as shuffles of M have mass uniformly spread on line segments, so each component (or portion of a component) of the support of C makes a linear contribution to V_C as it enters or leaves the family of squares $[t, t + 1/2]^2$. When C is an n-regular shuffle of M with n even, then V_C is linear on each subinterval of $P_n/2$ and the slopes of V_C can only change at an odd multiple of $1/2n$ if a fixed strip is flipped or if the support of C in one strip intersects either of the line segments between (0,1/2) and (1/2,1/2) or between (1/2,0) and (1,1/2) (which are part of the boundary of the hexagon $\bigcup_{t\in[0,1/2]} [t, t+1/2]^2$) at its midpoint. The conditions in the second part of this proposition preclude either of these from happening. □

A function $g : I \to I$ is a *Dyck path* (or, more specifically, an *n-Dyck path*) if g is piecewise linear on I with slope 1 or -1 on each subinterval of P_n and $g(0) = g(1) = 0$. Note that each Dyck path g satisfies the conditions on $\hat{\delta}$ in Proposition 2 and thus $t - g(t)$ is a diagonal.

Proposition 4. *If $C = M(P_n, \pi, \omega)$ is a shuffle of M for which $\pi^2 = e$ (the identity permutation) and π has no fixed points, then $\hat{\delta}_C$ is an n-Dyck path.*

Proof. For each pair (i, j) with $i < \pi(i) = j$, $\hat{\delta}_C$ has slope 1 on the i-th subinterval of P_n and slope -1 on the j-th subinterval of P_n. □

For each diagonal δ, the *associated Bertino copula* (see Bertino 1977 and Fredricks and Nelsen 2002) is defined by

$$B_\delta(u,v) = \min(u,v) - \min_{s\in[\{u,v\}]} \hat{\delta}(s),$$

where $[\{u,v\}]$ is the closed interval from $\min(u,v)$ to $\max(u,v)$. Note that the diagonal section of B_δ is δ. A *Bertino set* is a closed subset S of I^2 which is symmetric with respect to the ascending diagonal Δ of I^2, consists of the union of Δ and a collection of graphs of continuous, strictly decreasing functions, and satisfies the property:

$$\textit{if } (u,v) \in S \textit{ and } u < v, \textit{ then } S \cap (u,v) \times (v,1) \textit{ is empty.} \tag{2}$$

Proposition 5. *$M(P_n, \pi, \omega)$ is a Bertino copula if and only if $\pi^2 = e$, $w(i) = -1$ whenever i is not a fixed point of π, and*

$$\begin{aligned} &\textit{if } i, j \in \mathbf{n} \textit{ and } j \textit{ is between } i \textit{ and } \pi(i), \\ &\textit{then } \pi(j) \textit{ is also between } i \textit{ and } \pi(i). \end{aligned} \tag{3}$$

Proof. The support of $M(P_n, \pi, \omega)$ lies in a Bertino set if and only if each moved strip interchanges places with one other strip and is flipped, and (3) holds (as it is equivalent to (2) holding in this setting). The result follows from the fact (see Fredricks and Nelsen 2002) that a copula is a Bertino copula if and only if its support lies in a Bertino set. □

2. Critical shuffles

For each permutation π on $\mathbf{n}$, let M_π denote the flipped, n-regular shuffle of M determined by π. Such a shuffle M_π is *critical* (or, more specifically *n-critical*) if $\pi^2 = e$, π has no fixed points and (3) holds. Every critical shuffle is a Bertino copula by Proposition 5. For example, $M_{(1,6)(2,3)(4,5)}$ is a critical shuffle and its support is the union of the descending diagonals of the (1,6), (6,1), (2,3), (3,2), (4,5), (5,4) squares in the standard partition of I^2 into closed squares of sidelength 1/6 – specifically, the (i,j) square is $[(i-1)/6, i/6] \times [(j-1)/6, j/6]$. Note that any permutation π on $\mathbf{n}$ giving rise to a critical shuffle can be written as a product of transpositions (a_k, b_k), where each element of $\mathbf{n}$ appears once among the $a_k s$ and $b_k s$ and each pair (a_k, b_k) consists of an even and an odd integer.

Proposition 6. *If $C = M_\pi$ is an an n-critical shuffle, then $\hat{\delta}_C$ is an n-Dyck path. Conversely, if $\hat{\delta}$ is an n-Dyck path, then B_δ is an n-critical shuffle.*

Proof. The first part follows from Proposition 4. For the second part, note that n is even and that there are $n/2$ subintervals of P_n on which $\hat{\delta}$ has slope 1 and $n/2$ on which $\hat{\delta}$ has slope -1. If $\hat{\delta}$ has slope 1 on a subinterval J of P_n, then the function $h : I \to I$ defined by

$$h(u) = \max\{s \geq u : \hat{\delta}(t) \geq \hat{\delta}(u) \textit{ for all } t \in [u,s]\}$$

has slope -1 on J. It follows from Theorem 2.2 of Fredricks and Nelsen 2002 and $\hat{\delta}(t) = V_{B_\delta}([0,t] \times [t,1])$ that B_δ spreads mass $1/n$ uniformly on the graph of h restricted to J. Hence B_δ spreads no mass on Δ_J (the part of Δ corresponding to J). Since B_δ is symmetric, it spreads mass $1/n$ uniformly on the subset of $h(J) \times I$ which is the reflection about Δ of the graph of h restricted to J, and it spreads no mass on $\Delta_{h(J)}$. Since $h(J)$ is a subinterval of P_n on which $\hat{\delta}$ has slope -1 and the mass distribution on $h(J) \times I$ has already been determined, it is clear that B_δ is a flipped, n-regular shuffle of M and that the associated permutation has no fixed points. □

It is interesting to note that the natural correspondence between critical shuffles and Dyck paths in the preceding proposition leads to the conclusion that the number of n-critical shuffles is a Catalan number (see Stanley 1999, p. 221). The following theorem is reminiscent of the well-known fact (see Mikusinski et al 1992) that the shuffles of M are dense in the space of copulas.

Theorem 7. *The set of critical shuffles is dense in the space of Bertino copulas.*

Proof. Note that if δ_1 and δ_2 are diagonals that satisfy $\left|\hat{\delta}_1(t) - \hat{\delta}_2(t)\right| \leq \varepsilon$ for all $t \in I$, then $\left|\min_{t\in J} \hat{\delta}_1(t) - \min_{t\in J} \hat{\delta}_2(t)\right| \leq \varepsilon$ for any subinterval J of I, and hence

$$|B_{\delta_1}(u,v) - B_{\delta_2}(u,v)| = \left|\min_{t\in[\{u,v\}]} \hat{\delta}_1(t) - \min_{t\in[\{u,v\}]} \hat{\delta}_2(t)\right| \leq \varepsilon \text{ for all } (u,v) \in I^2.$$

Let δ be a diagonal and let $\epsilon > 0$ be given. Choose an even integer n for which $1/n \leq \epsilon$. Define $\hat{\delta}_\#$ inductively to be the piecewise linear function on I for which $\hat{\delta}_\#(0) = 0$ and, for each $k = 0, \ldots, n-1$, $\hat{\delta}_\#$ has slope 1 or -1 on $[k/n, (k+1)/n]$ depending on whether $\hat{\delta}((k+1)/n)$ is $\geq$ or $<$ $\hat{\delta}_\#(k/n)$, respectively. Then $\hat{\delta}_\#$ is a Dyck path and $\left|\hat{\delta}(t) - \hat{\delta}_\#(t)\right| \leq 1/n \leq \varepsilon$ for all $t \in I$. It follows from Proposition 6 and the preceding note that $B_{\delta_\#}$ is critical shuffle M_π and that

$$|B_\delta(u,v) - M_\pi(u,v)| \leq 1/n \leq \varepsilon \text{ for all } (u,v) \in I^2,$$

as desired. □

It is obvious from the preceding proof that the set of n-critical shuffles, with n taking on all values in any subsequence of the even integers, is also dense in the space of Bertino copulas. In the next section, we consider ns which are multiples of 4.

3. Critical graphs

A *critical* (or, more specifically, an *n-critical*) *graph* is a graph in the plane with an even number of vertices at the points corresponding to $\mathbf{n}$ along a horizontal line and $n/2$ nonintersecting edges, each edge connecting two vertices and lying above the horizontal line (see Stanley 1999, p. 222). If G is a critical graph and π is the permutation defined by $\pi(a) = b$ if a and b are vertices of the same edge of G, then $\pi^2 = e$, π has no fixed points and (3) holds. Conversely, a permutation with these properties determines a critical graph. Thus, there is a natural one-to-one correspondence between critical graphs and critical shuffles and we may represent either by a "critical" permutation.

The *weight* $w(G)$ of a critical graph G is the sum of the weights of the edges of G, where the weights of edges are assigned according to the following rules (L for left, R for right, Z for zero and S for short span), with $w((a, b))$ denoting the weight of the edge connecting a and b, with $a < b$.

Rule L: $w((a, b)) = a$ if $a, b \in [1, n/2]$;
Rule R: $w((a, b)) = n - b$ if $a, b \in [n/2 + 1, n]$;
Rule Z: $w((a, b)) = 0$ if $b - a > n/2$; and
Rule S: $w((a, b)) = n/2 - (b - a)$ otherwise.

As an example, the critical graph $G = (1, 2)(3, 6)(4, 5)(7, 8)$ has respective edge weights 1,1,3,0 and therefore $w(G) = 5$.

Suppose now that M_π is an n-critical shuffle with n a multiple of 4. Note that the conditions in the second part of Proposition 3 are vacuous as π has no fixed points and the absolute value of the difference between an even and an odd integer cannot be even. It follows that $V_\pi(t) = V_{M_\pi}([t, t + 1/2]^2)$ is piecewise linear on $[0, 1/2]$ and has constant slope on each subinterval of Q_n. (In fact, one can easily show that the only possible slopes are 0, 2 and -2, as the components of the support of M_π enter and exit the family of squares $[t, t + 1/2]^2$ in pairs.) Since the Trapezoidal Rule with $\Delta t = 1/n$ gives the exact value of integral, we conclude that

$$F(M_\pi) = [V_\pi(0) + 2V_\pi(1/n) + \ldots + 2V_\pi((n - 2)/2n) + V_\pi(1/2)]/2n.$$

Note that $V_\pi(k/n) = 2\eta(\pi)_k/n$ for $k = 0, \ldots, n/2$, where $\eta(\pi)_k$ is the number of pairs of interchanged strips that come from the original $k + 1$ through $k+n/2$ strips. Note that $\eta(\pi)_0 = \eta(\pi)_{n/2}$, as if π has m pairs of interchanges among the first half of strips, then it has $n/2 - 2m$ pairs of interchanges with one from the first half and one from the second half, and thus it must also have m pairs of interchanges from the second half of strips. Therefore $V_\pi(0) = V_\pi(1/2)$ and we have

$$F(M_\pi) = [V_\pi(0) + V_\pi(1/n) + \ldots + V_\pi((n - 2)/2n)]/n.$$

Suppose now that (a, b) is a transposition in π. The contribution of (a, b) to the bracketed sum in the preceding equation is $2\ell/n$, where ℓ is the number of intervals of the form $[k+1, k+n/2]$ for $k = 0, \ldots, n/2-1$ to which both a and b belong. If we assume that $a < b$, then $\ell = a$ if $a, b \in [1, n/2]$; $\ell = n - b$ if $a, b \in [n/2 + 1, n]$; $\ell = 0$ if $b - a > n/2$; and $\ell = n/2 - (b - a)$ otherwise. For example, if $\pi = (1, 10)(2, 3)(4, 9)(5, 6)(7, 8)(11, 12)$, then the values of ℓ for the transpositions in π in the displayed order are 0,2,1,5,4,0. Since these are the rules for computing edge weights of critical graphs we have

Theorem 8. *If M_π is an n-critical shuffle with n a multiple of 4 and G is the corresponding critical graph, then* $F(M_\pi) = 2w(G)/n^2$.

If n is not a multiple of 4, the preceding conclusion is invalid under the given weighting system.

4. Proof of the conjecture

Assume throughout this section that n is a fixed multiple of 4. The primary role in the proof of the conjecture is played by the critical graph $R = (1, n)(2, n-1) \ldots (n/2, n/2+1)$, called the *rainbow graph*, as it corresponds to the critical shuffle W. Note that the edge weights of R consist of $n/4$ zeros and the first $n/4$ odd integers, so $w(R) = n^2/16$, as the sum of the first k odd integers is k^2. It follows from the continuity of F (Proposition 1), the denseness of the critical shuffles (Theorem 7 and the remark at the end of section 2), the one-to-one correspondence between critical shuffles and graphs, and the formula in Theorem 8 that the validity of the Bertino conjecture is a consequence of

Theorem 9. *If G is an n-critical graph, then* $w(G) \geq w(R) = n^2/16$.

In the case $n = 4$, there are two critical graphs: (1,2)(3,4) and $R = (1, 4)(2, 3)$. The first has weight $1 + 0 = 1$; the second has weight $0 + 1 = 1$. In the case $n = 8$, there are 14 critical graphs. They are given along with their respective edge weights and their (total) weight in Table 1. Note that R appears first and that Theorem 9 is valid in the cases $n = 4$ and 8. Henceforth assume that n is a multiple of 4 which is ≥ 8.

Note from Table 1 that the weight of a critical graph seems to be less as more "nesting" occurs. This nesting feature is studied through the concept of rainbows. For each pair (k, r) of positive integers with $r \leq n/2$ and $k + 2r - 1 \leq n$, define the *rainbow*

$$R(k, r) = (k, k + 2r - 1)(k + 1, k + 2r - 2) \ldots (k + r - 1, k + r)$$

which consists of r edges with vertices all the integers in $[k, k + 2r - 1]$. The rainbow graph in this notation is $R(1, n/2)$. For each triple (k, r, s) of positive integers with $r + s \leq n/2$ and $k + 2(r + s) - 1 \leq n$, define the *double*

G	edge weights	w(G)
(1,8)(2,7)(3,6)(4,5)	0,0,1,3	4
(1,6)(2,5)(3,4)(7,8)	0,1,3,0	4
(1,4)(2,3)(5,8)(6,7)	1,2,0,1	4
(1,2)(3,8)(4,7)(5,6)	1,0,1,2	4
(1,2)(3,8)(4,5)(6,7)	1,0,3,1	5
(1,2)(3,6)(4,5)(7,8)	1,1,3,0	5
(1,6)(2,3)(4,5)(7,8)	0,2,3,0	5
(1,8)(2,7)(3,4)(5,6)	0,0,3,2	5
(1,8)(2,5)(3,4)(6,7)	0,1,3,1	5
(1,8)(2,3)(4,7)(5,6)	0,2,1,2	5
(1,2)(3,4)(5,8)(6,7)	1,3,0,1	5
(1,4)(2,3)(5,6)(7,8)	1,2,2,0	5
(1,8)(2,3)(4,5)(6,7)	0,2,3,1	6
(1,2)(3,4)(5,6)(7,8)	1,3,2,0	6

TABLE 1. Weights of critical graphs when $n = 8$.

rainbow $R(k, r, s) = R(k, r)R(k + 2r, s)$, which consists of $r + s$ edges with vertices all the integers in $[k, k + 2(r + s) - 1]$.

For each graph G which does not have n as a vertex, let G' denote the graph in which every edge of G is shifted one unit to the right. The difference in the weights of an edge E and its shift E' depends only on the rules used to compute the weights of those edges. The notation $X \to Y$ means that Rule X applies to E, while Rule Y applies to E'.

Proposition 10. *Let $E = (a, b)$ with $a + b$ odd and $b < n$. Then $w(E') - w(E)$ is 1 if $L \to L$; 0 if $L \to S$, $S \to S$ or $Z \to Z$; and -1 if $S \to R$ or $R \to R$.*

Proof. If $L \to S$, then $b = n/2$ and $w(E') = a = w(E)$. If $S \to R$, then $a = n/2$ and $w(E) = n - b$, while $w(E') = n - b - 1$. The other parts are obvious. □

Proposition 11. *For each $t \in [1, n/2]$, $w(R(1, t))$ is the sum of the first t entries in the list $1, 2, \ldots, n/4, n/4 - 1, \ldots, 1, 0$. Consequently, the sum of the whole list is $n^2/16$ and the weight of $R(1, n/2 - 1)$ is $n^2/16$.*

Proof. If $t \in [1, n/4]$, then Rule L applies to all of the edges of $R(1, t)$ and thus $w(R(1, t))$ is the sum of the first t positive integers. Let $f(k) = w(R(1, n/4 + k)) - w(R(1, n/4 + k - 1))$ for each $k \in [1, n/4]$. When $k = 1$, note that $R(1, n/4 + 1) = (1, n/2 + 2)R'(1, n/4)$ and hence $w(R(1, n/4 + 1)) = w(R'(1, n/4))$ as Rule Z applies to the edge $(1, n/2 + 2)$. Therefore

$f(1) = n/4 - 1$ by Proposition 10, as one edge is $L \to S$ and $n/4 - 1$ edges are $L \to L$. When $k = 2$, we have $w(R(1, n/4 + 2)) = w(R'(1, n/4 + 1))$ and hence $f(2) = n/4 - 2$ by Proposition 10 as one edge is $Z \to Z$, one is $S \to S$, one is $L \to S$ and $n/4 - 2$ are $L \to L$. Since Rule R is never used and with each successive k there is one fewer $L \to L$ edge, we see from Proposition 10 that $f(k) = n/4 - k$ and the first part is established. The second part follows from the fact that $R(1, n/2)$ is the rainbow graph and that the last entry in the list is 0. □

A similar analysis of rule application leads to a technique for computing the weight of shifted rainbows. Consider the case $n = 20$. When $t = 4$ we have the following rule applications, where, in each set of four, the edges from left to right are in increasing order of the smallest vertex of each edge.

$$LLLL \to LLLL \to LLLL \to SLLL \to SSLL \to SSSL \to SSSS \to$$
$$SSSR \to SSRR \to SRRR \to RRRR \to RRRR \to RRRR$$

In the case $t = 7$, the rule applications using the ordering of the smallest vertex of each edge are

$$ZZSSLLL \to ZZSSSLL \to ZZSSSSL \to ZZSSSSS \to$$
$$ZZSSSSR \to ZZSSSRR \to ZZSSRRR.$$

Any Zs that occur at the start (for $w(R(1, t))$) will continue at each successive step and thus may be ignored. Using Proposition 10, we arrive at Table 2, which specifies every value for $w(R(1, t))$ and each of its incremental shifts in the case $n = 20$.

t	w(R(1,t))	Shift list: t_j
1	1	1,1,1,1,1,1,1,1,0,-1,-1,-1,-1,-1,-1,-1,-1,-1
2	3	2,2,2,2,2,2,1,0,-1,-2,-2,-2,-2,-2,-2,-2
3	6	3,3,3,3,2,1,0,-1,-2,-3,-3,-3,-3,-3
4	10	4,4,3,2,1,0,-1,-2,-3,-4,-4,-4
5	15	4,3,2,1,0,-1,-2,-3,-4,-5
6	19	3,2,1,0,-1,-2,-3,-4
7	22	2,1,0,-1,-2,-3
8	24	1,0,-1,-2
9	25	0,-1
10	25	NA

TABLE 2. The values of $w(R(k, t))$ when $n = 20$.

For example, $w(R(3, 5)) = 15 + 4 + 3 = 22$ as $R(3, 5)$ is $R(1, 5)$ shifted twice to the right. It is easy to see that the obvious pattern in the case $n = 20$ works in general. Thus we have

Proposition 12. *Suppose* $1 \leq t < n/2$. *The shift list for* t *has* $n - 2t$ *entries as follows. If* $t < n/4$, *then the first half of the list from right to left is* $0, 1, \ldots, t$, *then remaining* t *for a total of* $n/2 - 2t$ t*'s, and the second half of the list from left to right is* $-1, -2, \ldots, -t$, *then remaining* $-t$ *for a total of* $n/2 - 2t + 1$ $-t$*'s. If* $n/4 \leq t < n/2$, *then the first half of the list from right to left starts with 0 and increases by 1 at each step, and the second half of the list from left to right starts with* -1 *and decreases by 1 at each step.*

We now consider the double rainbows. Since $R(1, r, s) = R(1, r)\ R(2r + 1, s)$, we see that $w(R(1, r, s)) = w(R(1, r)) + w(R(1, s)) + S(2r, s)$, where $S(m, t)$ is the sum of the first m entries in the shift list for t. Note from Table 2 that in the case $n = 20$, $w(R(1, 6, 2)) = 19 + 3 + 6 = 28$. We can also compute weights of double rainbows that don't have 1 as a vertex. For example, $w(R(2, 6, 2)) = w(R(1, 6, 2)) + 6_1 + 2_{13} = 28 + 3 - 2 = 29$, as $R(2, 6, 2)$ is obtained from $R(1, 6, 2)$ by shifting both $R(1, 6)$ and $R(13, 2)$ one unit to the right.

We will use properties of shift lists to establish results about the comparative weights of rainbows and double rainbows. For each pair of positive integers (r, s) with $r + s \leq n/2$, let $\mu(r, s) = w(R(1, r, s)) - w(R(1, r + s))$.

Proposition 13. *Let (r,s) be a pair of positive integers with* $r + s \leq n/2$.

(a) *If* $r + s = n/2$, *then* $\mu(r, s) = 0$ *and, consequently,* $w(R(1, r, s)) = w(R)$.
(b) *If* $r + s \leq n/4$, *then* $\mu(r, s) = rs$.
(c) *If* $r \geq n/4$, $s < n/4$ *and* $r + s < n/2$, *then* $\mu(r, s) - \mu(r + 1, s) = s$.
(d) *If* $r < n/4$, $s \geq n/4$ *and* $r + s < n/2$, *then* $\mu(r, s) - \mu(r, s + 1) = r$.
(e) *If* $r < n/4$ *and* $n/4 - r < s < n/4$, *then* $\mu(r, s) - \mu(r, s - 1) = n/2 - r - 2s + 1$.

Consequently, μ *is symmetric (and so* $w(R(1, r, s)) = w(R(1, s, r))$ *whenever* $r + s \leq n/2$*), and* μ *is nonnegative.*

Proof. For (a), the vertices of $R(1, r, s)$ are all of $\mathbf{n}$ and $R(1, r + s)$ is the rainbow graph. Suppose $s \geq n/4$. Since the sum of the whole shift list for s is $-r$, the weight of $R(1, s)$ shifted all the way to the right is $w(R(1, s)) - r$, which by Proposition 11 is $n^2/16$ minus the sum of the first r entries in the list of consecutive integers from 1 to $n/4$. Since the weight of $R(1, r)$ is the sum of the first r entries in the same list, we see that $w(R(1, r, s)) = n^2/16 = w(R)$ and hence $\mu(r, s) = 0$. If $s \leq n/4$, then the sum of the whole shift list for s is $-s$ and the weight of $R(1, s)$ shifted all the way to the right is the sum of the first s entries in the list of consecutive integers from 0 to $n/4 - 1$. Since the weight of $R(1, r)$ is $r^2/16$ minus the sum of the first s entries in the same list, we again see that $\mu(r, s) = 0$. For (b), note that Rule L applies to all of the edges. Since $w(R(1, r, s))$ is the sum of the consecutive integers from 1 to r and those from $2r+1$ to $2r+s$, while $w(R(1,r+s))$ is the sum of the consecutive integers from 1 to $r+s$, the result is clear. For (c), note that $w(R(1,t+1)) = w(R'(1,t))$ whenever $t \geq n/4$ and therefore

$$\begin{aligned}
&\mu(r,s) - \mu(r+1,s) \\
&\quad = w(R(1,r,s)) - w(R(1,r+s)) - \big[w(R(1,r+1,s) \\
&\qquad - w(R(1,r+s+1))\big] \\
&\quad = w(R(1,r)) + w(R(2r+1,s)) - w(R(1,r+s)) \\
&\qquad - \big[w(R(1,r+1)) + w(R(2r+3,s)) - w(R(1,r+s+1))\big] \\
&\quad = w(R(1,r+s+1)) - w(R(1,r+s)) - \big[w(R(1,r+1)) \\
&\qquad - w(R(1,r)) + w(R(2r+3,s)) - w(R(2r+1,s))\big] \\
&\quad = w(R'(1,r+s)) - w(R(1,r+s)) - \big[w(R'(1,r)) \\
&\qquad - w(R(1,r)) + w(R''(2r+1,s) - w(R(2r+1,s))\big] \\
&\quad = (r+s)_1 - r_1 - s_{2r+1} - s_{2r+2} = s,
\end{aligned}$$

as $t_1 = n/2 - 1 - t$ whenever $t \geq n/4$ and $s_j = -s$ whenver $j \geq n/2$. For (d), we have

$$\begin{aligned}
\mu(r,s) - \mu(r,s+1) &= w(R(1,r,s)) - w(R(1,r+s)) \\
&\qquad - [w(R(1,r,s+1)) - w(R(1,r+s+1))] \\
&= w(R(1,r+s+1)) - w(R(1,r+s)) \\
&\qquad - [w(R(2r+1,s+1)) - w(R(2r+1,s))] \\
&= w(R'(1,r+s)) - w(R(1,r+s)) \\
&\qquad - \big[w(R'(2r+1,s)) - w(R(2r+1,s))\big] \\
&= (r+s)_1 - s_{2r+1}.
\end{aligned}$$

The result follows from the consequence of Proposition 12 that the first shift entry in a line is 1 more than the third entry in the preceding line ($r=1$); that the first shift entry in a line is 2 more than the fifth entry two lines above ($r=2$); etc. For (e), note that $r \neq 1$ and $s \neq 1$. Also $2r+1 < n/2$, $2r+2s > n/2$ and $2s-1 < n/2$, so $w((2r+1, 2r+2s)) = n/2 - 2s + 1$ by Rule S. Now

$$\begin{aligned}
\mu(r,s) - \mu(r,s-1) &= w(R(1,r,s)) - w(R(1,r+s)) \\
&\quad - [w(R(1,r,s-1)) - w(R(1,r+s-1))] \\
&= w(R(2r+1,s)) - w(R(2r+1,s-1)) \\
&\quad - [w(R(1,r+s)) - w(R(1,r+s-1))]
\end{aligned}$$

$$= w((2r+1, 2r+2s)) + w(R'(2r+1, s-1)) - w(R(2r+1, s-1)) \\ - \big[w(R'(1, r+s-1)) - w(R(1, r+s-1))\big] \\ = n/2 - 2s + 1 + (s-1)_{2r+1} - (r+s-1)_1.$$

Fix r throughout the remainder of the proof. Let $\bar{s} = n/4 - r + 1$, which is the minimal applicable s for part (d). Two consequences of Proposition 12 are

$$(\bar{s}-1)_{2r+1} = (n/4 - r)_{2r+1} = n/4 - r - 1$$

and $(r + \bar{s} - 1)_1 = (n/4)_1 = n/4 - 1$ so $(\bar{s}-1)_{2r+1} - (r+\bar{s}-1)_1 = -r$ and the result holds for $\bar{s}$. Finally, the lists $\{(s-1)_{2r+1}\}$ and $\{(r+s-1)_1\}$ decrease by 1 for applicable s's, so the list $\{\mu(r,s) - \mu(r,s-1)\}$ decreases by 2 for applicable s's and (d) follows. Parts (a) and (b) give the values of μ on the line $r + s = n/2$ and the triangle $r + s \leq n/4$, respectively. Parts (c) and (a) determine the values of μ from right to left on the triangle $n/4 \leq r < n/2 - 1$, $1 \leq s < n/2 - 1$. Parts (d) and (a) determine the values of μ from top to bottom on the triangle $1 \leq r < n/4$, $n/4 \leq s < n/2 - r$. The remaining values of μ are given from bottom to top by parts (e) and (b). It is not difficult to see that the pattern in Table 3 for the case $n = 20$ holds in general and hence that μ is nonnegative and symmetric. □

9	0								
8	1	0							
7	2	2	0						
6	3	4	3	0					
5	4	6	6	4	0				
4	4	7	8	7	4	0			
3	3	6	8	8	6	3	0		
2	2	4	6	7	6	4	2	0	
1	1	2	3	4	4	3	2	1	0
s/r	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**

TABLE 3. Values of μ when $n = 20$.

Note that the entries in the subtriangle $r + s \leq n/4$ are repeated in the two subtriangles corresponding to parts (c) and (d) in the preceding proposition. Also note that the entries in the remaining central subtriangle are larger than the entries in the other three subtriangles.

We begin an examination of $w(R(k,r,s)) - w(R(k, r+s))$ by returning to the shift lists in the case $n = 20$. As an example, take $r = 3$ and $s = 1$. As usual,

$$w(R(1,3,1)) = w(R(1,3)) + w(R(1,1)) + S(6,1) = 6 + 1 + 6 = 13.$$

Now $w(R(2,3,1))$ is the sum of $w(R(1,3,1))$, the seventh entry in the first shift list and the first entry in the third shift list, i.e., $w(R(2,3,1)) = 13 + 1 + 3 = 17$. From the fourth shift list, note that $w(R(2,4)) = w(R(1,4)) + 4$. Therefore

$$w(R(2,3,1)) - w(R(2,4)) = w(R(1,3,1)) - w(R(1,4)).$$

In fact, if one deletes the first six entries in the first shift list and adds that new list to the first part of the third shift list, one obtains the fourth shift list. Thus, $w(R(k,3,1)) - w(R(k,4))$ is independent of k (in this case for $k \in [1,12]$). One can easily check in the case $n = 20$ that for any pair of positive integers (r,s) with $r + s < 10$, the sum of the s list with the first $2r$ entries deleted and the first part of the r list is equal to the $r + s$ list. A consequence of Proposition 12 is that this property holds in shift lists for any n. Thus, we have

Proposition 14. *For each triple (k,r,s) of positive integers with $r + s \leq n/2$ and $k + 2(r+s) \leq n+1$, $w(R(k,r,s)) - w(R(k,r+s)) = \mu(r,s)$ and, consequently, $w(R(k,r,s)) = w(R(k,s,r))$.*

It follows from Propositions 13 and 14 that replacing a double rainbow with a rainbow in the same place will yield a new graph that weighs less than the original graph, except in the case that a double rainbow is replaced by the rainbow graph and then the graphs will weigh the same. More generally, we may replace any number of adjacent rainbows with a rainbow in the same place without increasing the weight of the original graph.

To prove Theorem 9, let G be any n-critical graph. If every edge of G belongs to a rainbow, then G is either the rainbow graph or G consists of adjacent rainbows, and hence $w(G) \geq w(R)$. Now suppose that G has at least one edge that does not belong to a rainbow and let E be such an edge of G of minimal distance between its vertices. Then all the edges of G "inside" E form a system of adjacent rainbows and replacing them with a rainbow in the same place yields a graph that weighs less than G and for which the edge E is part of a rainbow. Continuing this process will eventually lead to the rainbow graph R and thus $w(G) \geq w(R)$. This completes the proof of Theorem 9 and establishes the validity of the Bertino conjecture.

Note that a consequence of Proposition 13(a) is that there are an infinite number of critical shuffles M_π for which $F(M_\pi) = 1/8$. Each of these shuffles is either W or has support which is the union of two line segments.

Acknowledgements. The author would like to thank Naiomi Cameron for suggesting the use of graphs. Additional thanks go to Michael Taylor, Carlo Sempi and Violetta Piperigou for their warm hospitality during the completion of this work.

References

Bertino, S. (1977), *Sulla dissomiglianza tra mutabili cicliche*, Metron **35**, 53–88.

Fredricks, G. A., and Nelsen, R. B. (1997), *Copulas constructed from diagonal sections*; In: Distributions with Given Marginals and Moment Problems (Eds. V. Beneš, and J. Štěpán), Kluwer, Dordrecht, pp. 129–136.

Fredricks, G. A., and Nelsen, R. B. (2002), *The Bertino family of copulas*; In: Distributions with Given Marginals and Statistical Modelling (Eds. C. M. Cuadras, J. Fortiana, and J. A. Rodríguez Lallena), Kluwer, Dordrecht, pp. 81–92.

Mikusinski, P., Sherwood, H., and Taylor, M. D. (1992), *Shuffles of Min*, Stochastica **13**, 61–74.

Nelsen, R. B. (2006), *An Introduction to Copulas*, Second Edition, Springer Series in Statistics, Springer, New York.

Stanley, R. P. (1999), *Enumerative Combinatorics. Volume 2*, Cambridge Studies in Advanced Mathematics 62, Cambridge University Press, Cambridge.

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